

LOCALIZATION FOR $THH(ku)$ AND THE TOPOLOGICAL HOCHSCHILD AND CYCLIC HOMOLOGY OF WALDHAUSEN CATEGORIES

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ABSTRACT. We prove a conjecture of Hesselholt and Ausoni-Rognes, establishing localization cofiber sequences of spectra

$$THH(H\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU) \longrightarrow \Sigma THH(H\mathbb{Z})$$

and

$$TC(H\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow TC(ku|KU) \longrightarrow \Sigma TC(H\mathbb{Z})$$

for the topological Hochschild and cyclic homology (THH and TC) of topological K -theory. These sequences support Hesselholt's view of the map $\ell \rightarrow ku$ as a “tamely ramified” extension of ring spectra, and validate the hypotheses necessary for Ausoni's simplified computation of $V(1)_*K(KU)$.

In order to make sense of the relative term $THH(ku|KU)$ and prove these results, we develop a theory of THH and TC of Waldhausen categories and prove the analogues of Waldhausen's theorems for K -theory. We resolve the longstanding confusion about localization sequences in THH and TC , and establish a specialized dévissage theorem.

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Introduction

Algebraic K -theory provides a high-level invariant of the homotopy theory of categories with a notion of extension and equivalence. The component group, K_0 , is the universal target for Euler characteristics, and higher algebraic K -theory captures subtle information intricately tied to number theory and geometry. For the algebraic K -theory of rings, trace methods using topological Hochschild homology (THH) and topological cyclic homology (TC) have proved remarkably successful at making K -theory computations tractable via the methods of equivariant stable homotopy theory.

At first glance K -theory and THH take very different inputs and have very different formal properties. For algebraic K -theory, the input is typically a Waldhausen category: a category with subcategories of cofibrations and weak equivalences. For THH , the basic input is a spectral category: a category enriched in spectra. While THH shares K -theory's additivity properties, THH seems to lack K -theory's approximation and localization properties [7]. A specific example of this failure was studied at great length in the paper [13]. From the perspective of the algebraic K -theory of rings and connective ring spectra, where THH is the stabilization of K -theory, this discrepancy is in some ways surprising, as one might expect THH to inherit the fundamental properties of K -theory.

In this paper, we construct THH for a general class of Waldhausen categories, and show that much of the apparent mismatch of formal properties is a consequence of the former mismatch of input data. We obtain an analogue of Waldhausen's Approximation Theorem [28] for THH . On the other hand, we observe that THH has two different analogues of the localization sequence in Waldhausen K -theory (the "Fibration Theorem" [28]). One of the localization sequences for THH was developed in our companion paper on localization in THH of spectral categories [4]; when applied to the K -theory of schemes, this sequence produces an analogue of the localization sequence of Thomason-Trobaugh [26]. The other localization sequence generalizes the localization sequence of Hesselholt-Madsen [13]. One of the principal contributions of this paper is to provide a conceptual explanation of

the two localization sequences of THH in relation to the localization sequence of K -theory.

As we explain in Sections 2 and 14, a Waldhausen category that admits factorizations has two spectral categories associated to it, a connective and a non-connective variant. The non-connective theory is “correct”, from the perspective of abstract homotopy theory and satisfies localization for cofiber sequences of spectral categories [4], but the connective theory is more closely related to K -theory. We show that the two theories agree under connectivity hypotheses that we make explicit in Section 9; in particular, for rings and connective ring spectra both spectral categories produce the expected THH . For exact categories, the connective version agrees with the THH of exact categories defined by Dundas-McCarthy [8]. For categories of complexes, the non-connective version agrees with the THH of the spectral derived category studied in [4]. Working with the non-connective theory gives the Thomason-Trobaugh style localization sequences, and working with the connective theory gives the Hesselholt-Madsen style localization sequences.

As a main application of this theory, we prove the localization sequence associated to the transfer map from $H\mathbb{Z}$ to ku that was conjectured by Hesselholt and Ausoni-Rognes [1, 2, 3]. Specifically, we construct naturally out of the category of ku -modules a simplicial spectral category $W^\Gamma(ku|KU)$ and cofiber sequences in the stable category

$$THH(\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU) \longrightarrow \Sigma THH(\mathbb{Z})$$

and

$$TC(\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow TC(ku|KU) \longrightarrow \Sigma TC(\mathbb{Z}),$$

compatible via a trace map with the localization cofiber sequence in K -theory established in [5]. Corresponding results hold for the Adams summand in the p -local and p -complete cases; see Theorem 11.1 below for details. These localization sequences were conjectured by Hesselholt and Ausoni-Rognes to explain the relationship of the computations of $K(\ell)$ and $K(ku)$; they support the perspective that $\ell \rightarrow ku$ should be an example of a “tamely ramified” extension of ring spectra. Furthermore, using these localization sequences, one can dramatically simplify Ausoni’s computation of $K(ku)$ [2, 8.4] by mimicking the de Rham-Witt arguments in Hesselholt-Madsen [13]. These localization sequences provide the chromatic level 1 analogues of the chromatic level 0 sequence of Hesselholt and Madsen [13]. Another application of these localization sequences is to compute $K(KU)$. One would like to use Ausoni’s computations of $K(ku)$ along with the localization cofiber sequence

$$K(\mathbb{Z}) \longrightarrow K(ku) \longrightarrow K(KU) \longrightarrow \Sigma K(\mathbb{Z})$$

to evaluate $K(KU)$. The transfer map in this sequence is controlled by the behavior of the transfer map in the associated sequences in THH and TC , where it is easier to understand. Following Hesselholt, Ausoni [2, 8.3] observes that in light of his calculations, the existence of the localization cofiber sequence in THH along with an algebraic fact would permit the complete identification of $V(1)_*K(KU)$.

One of the interesting aspects in the construction of the localization sequences is the construction of the relative terms $THH(ku|KU)$ and $TC(ku|KU)$: these relative terms “mix” the weak equivalences in the category of ku -modules with the weak equivalences in the category of KU -modules, in a way which does not arise in algebraic K -theory. This mixing is the reason why there are two different

localization sequences. In order to explain these sorts of relative terms, Rognes [20] has developed a theory of log ring spectra motivated by the appearance of log rings in the work of Hesselholt and Madsen [13]. We expect that our relative terms agree with the log THH and TC defined by Rognes.

Because our primary interest is the construction and explanation of the localization sequences above, we have taken a technical shortcut that drastically simplifies the theory. In Section 1, we introduce the concept of a simplicially enriched Waldhausen category in which the Waldhausen structure and the simplicial mapping spaces satisfy strong consistency hypotheses. The motivating example of such a category is a subcategory of the cofibrant objects in a simplicial model category with all objects fibrant; the model structure on the module categories of [11] satisfy this condition. For the majority of the paper we work only with simplicially enriched Waldhausen categories. In Section 14, we argue that simplicially enriched Waldhausen categories are not unduly restrictive by showing that a closed Waldhausen subcategory of a Waldhausen category that admits factorization is equivalent to a simplicially enriched Waldhausen category (in fact, a simplicial model category where every object is fibrant). This equivalence is functorial up to a zigzag of natural weak equivalences.

Although we have taken Waldhausen categories for the basic input to THH and K -theory in this paper, alternatively, one could take quasi-categories as the basic input. At this stage, the quasi-category approach would require serious background treatment of the K -theory and THH of quasi-categories, which is not yet formalized in the literature. On the other hand, since our first step is to replace a general Waldhausen category with a stable simplicial model category, such a background treatment would be essentially independent of the main work in this paper.

In this paper, whenever we work with topological spaces, the reader should understand that we are working in the category of compactly generated weak Hausdorff spaces. We use the words “topological” or “topological space” to highlight when we are using topological spaces rather than simplicial sets; these words should not be construed to imply the use of general topological spaces rather than compactly generated weak Hausdorff spaces.

Part 1. THH and TC of simplicially enriched Waldhausen categories

1. SIMPLICIALLY ENRICHED WALDHAUSEN CATEGORIES

In this section we introduce the structure of a simplicially enriched Waldhausen category. This structure compatibly combines a simplicial enrichment with a Waldhausen structure in a way that we make precise in Definition 1.1. Although this structure suffices for us to define an associated spectral category in the next section, more conditions are necessary to ensure that the homotopy theory of the enrichment matches up with the intrinsic homotopy theory of the Waldhausen category; we make these conditions precise in the definition of DK-compatible enrichment in Definition 1.2. In practice, and as we explain in Section 14, without much loss of generality, we typically have the stronger structures that we describe in Definitions 1.6 and 1.8. We begin with the most basic structure in the following definition.

Definition 1.1. A *simplicially enriched Waldhausen category* consists of a category $\mathcal{C} = \mathcal{C}_\bullet$ enriched in simplicial sets together with a Waldhausen category structure on \mathcal{C}_0 such that:

- (i) The zero object $*$ in \mathcal{C}_0 is a zero object for \mathcal{C} ,
- (ii) Pushouts over cofibrations in \mathcal{C}_0 are pushouts in \mathcal{C} ,
- (iii) Cofibrations $x \rightarrow y$ induce Kan fibrations $\mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z)$ for all objects z , and
- (iv) A map $x \rightarrow y$ is a weak equivalence if and only if $\mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z)$ is a weak equivalence for all objects z if and only if $\mathcal{C}(z, x) \rightarrow \mathcal{C}(z, y)$ is a weak equivalence for all objects z .

An *enriched exact functor* between such categories is a simplicial functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ that restricts to an exact functor of Waldhausen categories $\mathcal{C}_0 \rightarrow \mathcal{D}_0$.

Since the initial map $*$ \rightarrow x is always a cofibration in a Waldhausen category, Definition 1.1 implies that all the mapping spaces $\mathcal{C}(x, y)$ are Kan complexes. The fact that weak equivalences are detected on the simplicial mapping spaces implies that weak equivalences in \mathcal{C}_0 are closed under retracts and satisfy the two out of three property.

As explained by Dwyer and Kan, any category with a subcategory of weak equivalences has an intrinsic homotopy theory in terms of a functorial simplicially enriched category called the Dwyer-Kan simplicial localization [10]. Technically, we will use exclusively the variant called the hammock localization [9], which we will denote by L . Then for a simplicial Waldhausen category \mathcal{C} , the Dwyer-Kan simplicial localization of the underlying category with weak equivalences $L\mathcal{C}_0$ provides a second simplicially enriched category expanding \mathcal{C}_0 . In general, we see no reason why these two simplicial enrichments should be equivalent; we therefore introduce the following terminology.

Definition 1.2. Let \mathcal{C} be a simplicially enriched Waldhausen category. We say that \mathcal{C} is *DK-compatible* if for all objects x, y in \mathcal{C} , the maps

$$\mathcal{C}(x, y) \longrightarrow \text{diag } LC_\bullet(x, y) \longleftarrow LC_0(x, y)$$

are weak equivalences of simplicial sets. Here we regard \mathcal{C}_n as a category with weak equivalences by declaring a map in \mathcal{C}_n to be a weak equivalence if and only if some (or, equivalently, every) iterated face map takes it to a weak equivalence in \mathcal{C}_0 .

For categories enriched in simplicial sets, spaces, or spectra, an enriched functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is called a *DK-embedding* when it induces a weak equivalence $\mathcal{C}(x, y) \rightarrow \mathcal{D}(\phi(x), \phi(y))$ for all objects x, y . A DK-embedding is a *DK-equivalence* when it induces an equivalence $\pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ on categories of components. On the other hand, for discrete categories \mathcal{C}_0 and \mathcal{D}_0 with subcategories of weak equivalences, a functor $\mathcal{C}_0 \rightarrow \mathcal{D}_0$ that preserves weak equivalences is called a *DK-embedding* or *DK-equivalence* when it induces one on the Dwyer-Kan simplicial localizations. The main purpose of the previous definition is the following easy observation.

Proposition 1.3. *Let \mathcal{C} and \mathcal{D} be simplicially enriched Waldhausen categories and $\phi: \mathcal{C} \rightarrow \mathcal{D}$ a simplicial functor (not necessarily exact). Then:*

- (i) $\phi_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ preserves weak equivalences.
- (ii) Assume furthermore that \mathcal{C} and \mathcal{D} are both DK-compatible. Then ϕ is a DK-embedding or DK-equivalence of simplicially enriched categories if and only if ϕ_0 is a DK-embedding or DK-equivalence (respectively) of categories with weak equivalences.

The following is an easy but important class of examples of DK-compatible simplicially enriched Waldhausen categories.

Example 1.4. An exact category, or more generally, a Waldhausen category whose weak equivalences are the isomorphisms becomes a DK-compatible simplicially enriched Waldhausen category by regarding its mapping sets as discrete simplicial sets.

We also have the following less trivial examples.

Example 1.5. Let \mathcal{C} be a Waldhausen subcategory of cofibrant objects in simplicial closed model category \mathcal{M} in which all objects are fibrant. Then \mathcal{C} is a simplicially enriched Waldhausen category with its natural simplicial mapping spaces and Waldhausen structure inherited from \mathcal{M} . If \mathcal{C} is closed under tensors with finite simplicial sets, then \mathcal{C} is a DK-compatible (see Theorem 1.9 below). Examples of this type include:

- (i) Finite cell R -modules for an EKMM S -algebra R , or (for R connective with π_0 noetherian) cell R -modules that have finite stage finitely generated Postnikov towers as in [5].
- (ii) The category of finite cell modules over a simplicial ring A , or the category of finite cell modules built out of finitely generated projective A -modules.
- (iii) The category of simplicial objects on an abelian category with the “split-exact” model structure (where the cofibrations are the levelwise split monomorphisms and the weak equivalences are the simplicial homotopy equivalences).
- (iv) The category of levelwise projectives in the category of simplicial objects on an abelian category with enough projectives (with the standard projective model structure). Likewise, the opposite category of the levelwise injectives in the category of cosimplicial objects on an abelian category with enough injectives (with the standard injective model structure).

In addition to being DK-compatible, the previous class of examples has an additional structure that we employ to construct non-connective spectral enrichments in the next section. We abstract this structure in the following definition.

Definition 1.6. A simplicially tensored Waldhausen category is a simplicially enriched Waldhausen category in which tensors with finite simplicial sets exist and satisfy the pushout-product axiom. A tensored exact functor between simplicially tensored Waldhausen categories is a enriched exact functor that preserves tensors with finite simplicial sets.

In the previous definition, the pushout-product axiom [24, 2.1] asserts that given a cofibration $x \rightarrow y$ in \mathcal{C}_0 and a cofibration $A \rightarrow B$ of finite simplicial sets, the map

$$(x \otimes B) \cup_{x \otimes A} (y \otimes A) \longrightarrow y \otimes B$$

is a cofibration in \mathcal{C}_0 . This axiom implies that the usual mapping cylinder construction endows \mathcal{C}_0 with a cylinder functor satisfying the cylinder axiom (in the sense of [28, §1.6]). The Kan condition on the mapping spaces combined with the tensor adjunction implies the following proposition.

Proposition 1.7. *Let \mathcal{C} be a simplicially tensored Waldhausen category.*

- (i) *For any object x in \mathcal{C} , the tensor $x \otimes (-)$ preserves weak equivalences in simplicial sets.*
- (ii) *For any finite simplicial set X , the tensor $(-) \otimes X$ preserves weak equivalences in \mathcal{C} .*

- (iii) For objects x and y in \mathcal{C} , the simplicial set $\mathcal{C}(x, y)$ is canonically isomorphic to $\mathcal{C}_0(x \otimes \Delta[\cdot], y)$.

Definition 1.6 provides the strongest background structure that we use; in Section 14, we see that Waldhausen categories quite generally admit equivalent models of this type. In our study of the localization sequence, however, we require slightly more flexibility. Using a simplicially tensored Waldhausen category as an ambient category, we will sometimes need to restrict to a subcategory.

Definition 1.8. A *enhanced simplicially enriched Waldhausen category* is a pair $\mathcal{A} \subset \mathcal{C}$ where \mathcal{C} is a simplicially tensored Waldhausen category and \mathcal{A} is a full subcategory such that \mathcal{A}_0 is a closed Waldhausen subcategory. For $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \subset \mathcal{D}$ enhanced simplicially enriched Waldhausen categories, an *enhanced exact functor* $\mathcal{A} \rightarrow \mathcal{B}$ is a tensored exact functor of simplicially tensored Waldhausen categories $\mathcal{C} \rightarrow \mathcal{D}$ that restricts to a functor $\mathcal{A} \rightarrow \mathcal{B}$.

As in [28, §1.2], a *Waldhausen subcategory* \mathcal{A} is a full subcategory of a Waldhausen category \mathcal{C} that itself becomes a Waldhausen category by taking a weak equivalence to be a weak equivalence in \mathcal{C} between objects of \mathcal{A} and a cofibration to be a cofibration in \mathcal{C} between objects of \mathcal{A} for which the cofiber is in \mathcal{A} (up to isomorphism). A *closed* Waldhausen subcategory is a Waldhausen subcategory $\mathcal{A} \subset \mathcal{C}$ that contains every object of \mathcal{C} that is weakly equivalent to an object of \mathcal{A} . An enhanced simplicially enriched Waldhausen category inherits tensors with homotopically trivial finite simplicial sets (but not necessarily arbitrary finite simplicial sets) as well as properties (i) and (iii) of Proposition 1.7. We also have the following compatibility result.

Theorem 1.9. *An enhanced simplicially enriched Waldhausen category $\mathcal{A} \subset \mathcal{C}$ is DK-compatible.*

Proof. Fix objects a, b . Regarding $\mathcal{A}_n(a, b)$ as $\mathcal{A}_0(a \otimes \Delta[n], b)$, each category \mathcal{A}_n admits a homotopy calculus of left fractions [9, 6.1] (see, for example, the argument for [6, 5.5]) and so we can replace $L\mathcal{A}_n(a, b)$ with the nerve of the category of words of the form $\mathbf{W}^{-1}\mathbf{C}$, which we will temporarily denote as $L_n(a, b)$. An object of this category consists of a zigzag

$$a \longrightarrow x \longleftarrow b$$

of maps in \mathcal{A}_n , where the map $x \leftarrow b$ is a weak equivalence; a map in this category is a map in \mathcal{A}_n of x that is under a and b . We check that both maps

$$L_0(a, b) \longrightarrow \text{diag } L_\bullet(a, b) \longleftarrow \mathcal{A}(a, b)$$

are weak equivalences.

For the map $L_0(a, b) \rightarrow \text{diag } L_\bullet(a, b)$, we show that each iterated degeneracy $L_0(a, b) \rightarrow L_n(a, b)$ is a weak equivalence. An iterated face map gives a functor $L_n(a, b) \rightarrow L_0(a, b)$ such that the composite is the identity on $L_0(a, b)$. To see that the composite on $L_n(a, b)$ is a weak equivalence, note that both inclusions of a and b in $a \otimes \Delta[1]$ and $b \otimes \Delta[1]$ induce weak equivalences

$$L_n(a \otimes \Delta[1], b \otimes \Delta[1]) \longrightarrow L_n(a, b).$$

Choosing a contracting homotopy $\Delta[1] \times \Delta[n] \rightarrow \Delta[n]$, we get a map from $L_n(a, b)$ to $L_n(a \otimes \Delta[1], b \otimes \Delta[1])$

$$(a \rightarrow x \leftarrow b) \longmapsto (a \otimes \Delta[1] \rightarrow x \otimes \Delta[1] \leftarrow b \otimes \Delta[1])$$

such that one composite on $L_n(a, b)$ is simplicially homotopic to the identity and the other is simplicially homotopic to the composite map $L_n \rightarrow L_0 \rightarrow L_n$ we are interested in.

It remains to see that the map $\mathcal{A}(a, b) \rightarrow \text{diag } L_\bullet(a, b)$ is a weak equivalence. We can identify $\text{diag } L_\bullet(a, b)$ as the diagonal of the bisimplicial set whose simplicial set of q -simplices is

$$\mathcal{A}(a, x_0) \times w\mathcal{A}(b, x_0) \times w\mathcal{A}(x_0, x_1) \times \cdots \times w\mathcal{A}(x_{q-1}, x_q),$$

where $w\mathcal{A}$ denotes the components with (any, or equivalently, all) vertices in $w\mathcal{A}_0$, the subcategory of weak equivalences of the Waldhausen category \mathcal{A}_0 . The map $\mathcal{A}(a, b) \rightarrow \text{diag } L_\bullet(a, b)$ factors through a bisimplicial map from the bisimplicial set $X_{\bullet\bullet}$ whose simplicial set of q -simplices $X_{q\bullet}$ is

$$\mathcal{A}(a, b) \times w\mathcal{A}(b, x_0) \times w\mathcal{A}(x_0, x_1) \times \cdots \times w\mathcal{A}(x_{q-1}, x_q).$$

The inclusion $\mathcal{A}(a, b) \rightarrow \text{diag } X_{\bullet\bullet}$ is clearly a simplicial homotopy equivalence, and the bisimplicial map $X_{\bullet\bullet} \rightarrow L_\bullet(a, b)$ is a degreewise weak equivalence. \square

2. SPECTRAL CATEGORIES ASSOCIATED TO SIMPLICIALLY ENRICHED WALDHAUSEN CATEGORIES

As we explain, a simplicially enriched Waldhausen category has an associated spectral category, which is natural in enriched exact functors. The mapping spectra in this category are prolongations of Γ -spaces, and as such, are always connective. For an enhanced simplicially enriched Waldhausen category, we can associate another spectral category, typically non-connective, using the suspensions in the ambient simplicially tensored Waldhausen category; it is natural in enhanced exact functors. In this section, we explore the basic properties of these categories. We begin with the construction.

Definition 2.1. Let \mathcal{C} be a simplicially enriched Waldhausen category. Define \mathcal{C}^Γ , the Γ -category associated to \mathcal{C} , to have objects the objects of \mathcal{C} and mapping Γ -spaces

$$\mathcal{C}_q^\Gamma(x, y) = \mathcal{C}(x, \bigvee_q y).$$

By abuse, we will also write \mathcal{C}^Γ for the enrichment in symmetric spectra obtained by prolongation. We will refer to \mathcal{C}^Γ as the *connective spectral enrichment* of \mathcal{C} or the connective spectral category associated to \mathcal{C} .

Here the composition

$$\mathcal{C}_r^\Gamma(y, z) \wedge \mathcal{C}_q^\Gamma(x, y) \longrightarrow \mathcal{C}_{qr}^\Gamma(x, z).$$

comes from the $\Sigma_q \wr \Sigma_r$ -equivariant map

$$\mathcal{C}(y, \bigvee_r z) \longrightarrow \prod_q \mathcal{C}(y, \bigvee_r z) \longrightarrow \mathcal{C}(\bigvee_q y, \bigvee_{rq} z)$$

and composition

$$\mathcal{C}(\bigvee_q y, \bigvee_{rq} z) \wedge \mathcal{C}(x, \bigvee_q y) \longrightarrow \mathcal{C}(x, \bigvee_{rq} z).$$

This composition of Γ -spaces then induces the composition on the associated symmetric spectra. The following proposition is immediate from the construction.

Proposition 2.2. *For simplicially enriched Waldhausen categories \mathcal{C} and \mathcal{D} , an enriched exact functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ induces a spectral functor $\phi^\Gamma: \mathcal{C}^\Gamma \rightarrow \mathcal{D}^\Gamma$. If \mathcal{C} and \mathcal{D} are DK-compatible and ϕ is a DK-embedding or DK-equivalence, then so is ϕ^Γ .*

In general, we can not expect the Γ -spaces $\mathcal{C}^\Gamma(x, y)$ to be special or very special. On the other hand, as a prolongation of a Γ -space, the associated symmetric spectrum is *semistable*: it represents the same object in the stable category as its underlying prespectrum.

Proposition 2.3. *The mapping symmetric spectra in \mathcal{C}^Γ are semistable.*

Example 2.4. For \mathfrak{E} be an exact category, simplicially enriched as in Example 1.4,

$$\mathfrak{E}_q^\Gamma(x, y) = \mathfrak{E}(x, \bigoplus_{i=1}^q y) \cong \prod_{i=1}^q \mathfrak{E}(x, y).$$

Prolonging to symmetric spectra, we get

$$\mathfrak{E}^\Gamma(x, y)(n) = \mathfrak{E}(x, y) \otimes \tilde{\mathbb{Z}}[S^n],$$

where $\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[*]$. This is precisely the spectral category associated to an exact category studied by Dundas-McCarthy [8] and Hesselholt-Madsen [13].

When \mathcal{C} is a simplicially tensored Waldhausen category, we can construct another enrichment in symmetric spectra using suspensions: for an object x in \mathcal{C} , let Σx be the cofiber of the map

$$x \otimes \partial\Delta[1] \longrightarrow x \otimes \Delta[1].$$

Suspension defines a tensored exact functor from \mathcal{C} to itself. Commuting colimits and tensors, and applying the associativity isomorphism for tensors, we can describe the iterated suspension $\Sigma^n x$ as the cofiber of the map

$$x \otimes \partial(\Delta[1]^n) \longrightarrow x \otimes \Delta[1]^n,$$

where $\Delta[1]^n = \Delta[1] \times \cdots \times \Delta[1]$. The n -th suspension inherits from $\Delta[1]^n$ an action of the symmetric group Σ_n .

Definition 2.5. Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category. Define \mathcal{A}^S be the spectral category with objects the objects of \mathcal{A} and mapping symmetric spectra

$$\mathcal{A}^S(x, y)(n) = \mathcal{C}(x, \Sigma^n y).$$

We will refer to this as the *non-connective spectral enrichment* of \mathcal{A} or the non-connective spectral category associated to \mathcal{A} .

In the previous definition, we obtain the composition on \mathcal{A}^S ,

$$\mathcal{A}^S(y, z) \wedge \mathcal{A}^S(x, y) \longrightarrow \mathcal{A}^S(x, z)$$

from the $\Sigma_n \times \Sigma_m$ -equivariant maps

$$\mathcal{C}(y, \Sigma^m z) \wedge \mathcal{C}(x, \Sigma^n y) \longrightarrow \mathcal{C}(\Sigma^n y, \Sigma^{m+n} z) \wedge \mathcal{C}(x, \Sigma^n y) \longrightarrow \mathcal{C}(x, \Sigma^{m+n} z).$$

Note that for a enhanced simplicially enriched Waldhausen category $\mathcal{A} \subset \mathcal{C}$, the suspension of an object of \mathcal{A} is an object of \mathcal{C} but need not be an object in \mathcal{A} . As a consequence, the non-connective enrichment \mathcal{A}^S depends strongly on the ambient simplicially tensored Waldhausen category \mathcal{C} . Recall that an enhanced exact functor has as part of its structure a tensored exact functor on the ambient simplicially

tensored Waldhausen categories; the following functoriality is immediate from the construction.

Proposition 2.6. *An enhanced exact functor $\phi: \mathcal{A} \rightarrow \mathcal{B}$ between enhanced simplicially enriched Waldhausen categories induces a spectral functor $\phi^S: \mathcal{A}^S \rightarrow \mathcal{B}^S$. If ϕ is a DK-equivalence and a DK-embedding on the ambient simplicially tensored category, then ϕ^S is a DK-equivalence.*

Using Proposition 1.7.(i) and the Kan condition, we see that the action of any even permutation on $\mathcal{A}^S(x, y)(n) = \mathcal{C}(x, \Sigma^n y)$ is homotopic to the identity. Then [22, 3.2] gives us the following proposition.

Proposition 2.7. *The mapping symmetric spectra in \mathcal{A}^S are semistable.*

Example 2.8. Let \mathfrak{A} be an abelian category with enough projectives (e.g., the opposite category of an abelian category with enough injectives), and let $\mathfrak{E} \subset \mathfrak{A}$ be an exact category (with exact sequences the exact sequences of \mathfrak{A} in \mathfrak{E}). Let \mathcal{C} be the simplicially tensored Waldhausen category of levelwise projectives in the category of simplicial objects of \mathfrak{A} , as in Example 1.5.(iv). Let $\mathcal{A} \subset \mathcal{C}$ be the full subcategory of \mathcal{C} consisting of those objects x such that $\pi_0 x$ is in \mathfrak{E} and $\pi_n x = 0$ for $n > 0$. Then $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicially enriched Waldhausen category and π_0 gives a enriched exact functor $\mathcal{A} \rightarrow \mathfrak{E}$. This functor induces a DK-equivalence of the connective spectral enrichments $\mathcal{A}^\Gamma \rightarrow \mathfrak{E}^\Gamma$. On the other hand \mathcal{A} has a non-connective spectral enrichment \mathcal{A}^S , where $\pi_n \mathcal{A}^S(x, y)$ is 0 for $n > 0$ and $\text{Ext}^{-n}(\pi_0 x, \pi_0 y)$ for $n \leq 0$.

Example 2.9. As an example to demonstrate the significance of the ambient simplicially tensored Waldhausen category, let \mathcal{C} be the Waldhausen category of countable cell EKMM S -modules and let \mathcal{C}' the Waldhausen category of countable cell EKMM $H\mathbb{Z}$ -modules (for some countable cell S -algebra model of $H\mathbb{Z}$). Let \mathcal{A} and \mathcal{A}' be the Waldhausen subcategories of Eilenberg-Mac Lane spectra with homotopy groups concentrated in degree zero in \mathcal{C} and \mathcal{C}' respectively. The forgetful functor $\mathcal{C}' \rightarrow \mathcal{C}$ is exact and sends \mathcal{A}' into \mathcal{A} , inducing a DK-equivalence and hence a DK-equivalence $\mathcal{A}'^\Gamma \rightarrow \mathcal{A}^\Gamma$ but not a DK-equivalence $\mathcal{A}'^S \rightarrow \mathcal{A}^S$.

For an enhanced simplicially enriched Waldhausen category $\mathcal{A} \subset \mathcal{C}$, we have a natural spectral functor (of spectral categories of symmetric spectra) $\mathcal{A}^\Gamma \rightarrow \mathcal{A}^S$. In the previous example, and in fact in the examples of Example 1.5, \mathcal{A}^Γ is a connective cover of \mathcal{A}^S , i.e., induces an isomorphism on the non-negative homotopy groups. The following proposition gives a sufficient general condition for this to hold.

Proposition 2.10. *Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category, and assume that for every $a, b \in \mathcal{A}$ the suspension map $\mathcal{C}(a, b) \rightarrow \mathcal{C}(\Sigma a, \Sigma b)$ is a weak equivalence. Then $\mathcal{A}^\Gamma(a, b) \rightarrow \mathcal{A}^S(a, b)$ is a connective cover.*

Proof. Fix a, b and consider the functor $F(-) = \mathcal{C}(a, (b \otimes -)/(b \otimes *))$ as a functor from based finite simplicial sets to based simplicial sets; we then get $\mathcal{A}^\Gamma(a, b)$ by viewing F as Γ -space and $\mathcal{A}^S(a, b)$ by viewing $\{F(S^n)\}$ as a symmetric spectrum. By the hypothesis of the proposition, the canonical map $F(-) \rightarrow \Omega F(\Sigma -)$ is a weak equivalence. The argument of [16, 17.9] shows that F is “linear” meaning that it takes homotopy pushouts to homotopy pullbacks, and in particular, as a Γ -space F is very special [16, 18.6]. The homotopy groups of $\mathcal{A}^\Gamma(a, b)$ are then the homotopy groups of $F(S^0) = \mathcal{A}(a, b)$. Likewise, $\{F(S^n)\}$ is an Ω -spectrum, so its non-negative homotopy are also the homotopy groups of $F(S^0)$. \square

In the absence of the stability hypothesis of the previous proposition, \mathcal{A}^S tends to better capture the stable homotopy theory of $\mathcal{A} \subset \mathcal{C}$, as indicated for example in the following proposition.

Proposition 2.11. *Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category.*

- (i) *For any x, y in \mathcal{A} , the map $\mathcal{A}^S(x, y) \rightarrow \mathcal{A}^S(\Sigma x, \Sigma y)$ is a weak equivalence.*
- (ii) *For a cofibration $f: a \rightarrow b$, Cf the homotopy cofiber, and any object z , the sequences*

$$\begin{aligned} \Omega \mathcal{A}^S(a, z) &\longrightarrow \mathcal{A}^S(Cf, z) \longrightarrow \mathcal{A}^S(b, z) \longrightarrow \mathcal{A}^S(a, z) \\ \mathcal{A}^S(z, a) &\longrightarrow \mathcal{A}^S(z, b) \longrightarrow \mathcal{A}^S(z, Cf) \longrightarrow \Sigma \mathcal{A}^S(z, a) \end{aligned}$$

form a fiber sequence and a cofiber sequence in the stable category, respectively.

Proof. Part (i) and the statement about the first sequence in part (ii) are clear. The statement about the second sequence follows from part (i) and the argument in [14, §III.2.1] or [16, 7.4.vi]. \square

The proposition indicates that for a simplicially tensored Waldhausen category \mathcal{C} , the spectral category \mathcal{C}^S is nearly “pretriangulated” in the sense of [4, 4.4]. In fact, we have the following easy corollary:

Corollary 2.12. *Let \mathcal{C} be a simplicially tensored Waldhausen category in which every object is weakly equivalent to a suspension. Then the category \mathcal{C}^S is pretriangulated, i.e., the category of components $\pi_0 \mathcal{C}^S$ has the structure of a triangulated category with triangles coming from the cofibration sequences and translation from the suspension.*

Remark 2.13. As the preceding results indicate, the construction of the mapping spectra described above provides a version of stabilization of the simplicial Waldhausen category \mathcal{C} . In particular, the zeroth space of (a fibrant replacement of) the mapping spectrum $\mathcal{C}^S(x, y)$ is given by

$$\operatorname{colim}_n \Omega^n \mathcal{C}(x, \Sigma^n y) \cong \operatorname{colim}_n \mathcal{C}(\Sigma^n x, \Sigma^n y).$$

It is possible to explicitly compare \mathcal{C}^S to a model of the formal stabilization in terms of symmetric spectrum objects in \mathcal{C} . We give an example below, but general theorems of this sort are encumbered with technical hypotheses, and since we do not need such results we leave this to the interested reader.

Example 2.14 (Spectral categories and stabilization in Waldhausen’s algebraic K -theory of spaces). Let G be a group-like topological monoid, let W a CW-complex on which G acts, and let $R(W, G)$ denote the category of G -spaces which have W as a retract. When restricting to objects satisfying some kind of finiteness condition, $R(W, G)$ provides Waldhausen’s motivating example for a Waldhausen category and one of the models underlying the algebraic K -theory of spaces. We can give $R(W, G)$ the model structure in which the weak equivalences are the equivariant maps that induce underlying equivalences of spaces. The category $R(W, G)$ is in no sense stable (for example, when G and W are trivial, $R(W, G)$ is the category of based spaces), and the spectral category $R(W, G)^S$ is equivalent to the evident subcategory of free $\Sigma_+^\infty G$ -spectra, as expected.

3. THE S_\bullet AND MOORE NERVE CONSTRUCTIONS

As part of the construction of THH and TC of simplicially enriched Waldhausen categories and the construction of the cyclotomic trace in the next section, we need to extend Waldhausen's S_\bullet construction and the nerve category construction to the context of simplicially enriched Waldhausen categories. We begin with the S_\bullet construction, where no difficulties arise.

Let $\text{Ar}[n]$ denote the lexicographically ordered set of ordered pairs of integers i, j where $0 \leq i \leq j \leq n$. Recall that for a Waldhausen category \mathcal{C}_0 , $S_n\mathcal{C}_0$ is the full subcategory of the category of functors $A = a_{-, -} : \text{Ar}[n] \rightarrow \mathcal{C}_0$ such that:

- (i) $a_{i,i} = *$,
- (ii) $a_{i,j} \rightarrow a_{i,k}$ is a cofibration, and
- (iii) $a_{i,i} \cup_{a_{i,j}} a_{i,k} \rightarrow a_{j,k}$ is an isomorphism

for all $i \leq j \leq k$. A map in $S_n\mathcal{C}_0$ is simply a natural transformation of functors $\text{Ar}[n] \rightarrow \mathcal{C}_0$. This becomes a Waldhausen category with weak equivalences defined objectwise and cofibrations defined to be the objectwise cofibrations $A \rightarrow B$ such that each map $a_{i,k} \cup_{a_{i,j}} b_{i,j} \rightarrow b_{i,k}$ is a cofibration.

Definition 3.1. For a simplicially enriched Waldhausen category \mathcal{C} , let $S_n\mathcal{C}$ be the simplicially enriched category with objects the same as $S_n\mathcal{C}_0$ and simplicial sets of maps the simplicial set of natural transformations of functors $\text{Ar}[n] \rightarrow \mathcal{C}$.

Condition (iii) in the definition of S_\bullet implies that a map $A \rightarrow B$ is completely determined by the maps $a_{0,j} \rightarrow b_{0,j}$. Since the maps $a_{0,j} \rightarrow a_{0,j+1}$ are cofibrations, we can identify the simplicial set of maps $S_n\mathcal{C}(A, B)$ as a pullback over fibrations

$$(3.2) \quad S_n\mathcal{C}(A, B) \cong \mathcal{C}(a_{0,1}, b_{0,1}) \times_{\mathcal{C}(a_{0,1}, b_{0,2})} \cdots \times_{\mathcal{C}(a_{0,n-1}, b_{0,n})} \mathcal{C}(a_{0,n}, b_{0,n}).$$

That is, the simplicial set of maps computes a homotopy limit. Using this formulation of the maps, the following becomes an easy check of the definitions and standard properties of pullbacks of fibrations of Kan complexes.

Proposition 3.3. *Let \mathcal{C} be a simplicially enriched Waldhausen category. Then:*

- (i) $S_n\mathcal{C}$ is a simplicially enriched Waldhausen category.
- (ii) If \mathcal{C} is simplicially tensored or enhanced, then so is $S_n\mathcal{C}$.
- (iii) The face and degeneracy maps $S_m\mathcal{C} \rightarrow S_n\mathcal{C}$ are enriched exact and are tensored exact or enhanced exact when \mathcal{C} is simplicially tensored or enhanced.

Moreover, S_n preserves enriched exact, tensored exact, and enhanced exact functors.

Applying the spectral category constructions of the previous section, we get a simplicial spectral category $S_\bullet\mathcal{C}^\Gamma$, natural in enriched exact functors of \mathcal{C} . When \mathcal{C} is simplicially tensored or enhanced, we get a simplicial spectral category $S_\bullet\mathcal{C}^S$, natural in tensored exact or enhanced exact functors of \mathcal{C} . The formula (3.2) for the mapping spaces then implies the following result for spectral categories.

Proposition 3.4. *Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be an enriched exact functor between simplicially enriched Waldhausen categories that are DK-compatible. If ϕ is a DK-embedding, then $S_n\phi^\Gamma: S_n\mathcal{C}^\Gamma \rightarrow S_n\mathcal{D}^\Gamma$ is a DK-embedding.*

Proposition 3.5. *Let $\phi: (\mathcal{A} \subset \mathcal{C}) \rightarrow (\mathcal{B} \subset \mathcal{D})$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories. If $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is a DK-embedding, then*

$$S_n\phi^S: S_n\mathcal{A}^S \longrightarrow S_n\mathcal{B}^S$$

is a DK-embedding.

In Proposition 3.4, we do not necessarily get a DK-equivalence $S_n \mathcal{C}^\Gamma \rightarrow S_n \mathcal{D}^\Gamma$ from a DK-equivalence $\mathcal{C} \rightarrow \mathcal{D}$. Applying the results of [6], we can do slightly better in Proposition 3.5.

Proposition 3.6. *Under the hypotheses of Proposition 3.5, if $\phi: \mathcal{A} \rightarrow \mathcal{B}$ and $\phi: \mathcal{C} \rightarrow \mathcal{D}$ are DK-equivalences, then*

$$S_n \phi^\Gamma: S_n \mathcal{A}^\Gamma \longrightarrow S_n \mathcal{B}^\Gamma \quad \text{and} \quad S_n \phi^S: S_n \mathcal{A}^S \longrightarrow S_n \mathcal{B}^S$$

are also DK-equivalences.

Proof. It suffices to show that for any sequence of cofibrations $b_1 \rightarrow \cdots \rightarrow b_n$ in \mathcal{B} , there exists a sequence of cofibrations $a_1 \rightarrow \cdots \rightarrow a_n$ in \mathcal{A} and a commutative diagram

$$\begin{array}{ccccccc} \phi(a_1) & \longrightarrow & \phi(a_2) & \longrightarrow & \cdots & \longrightarrow & \phi(a_{n-1}) & \longrightarrow & \phi(a_n) \\ \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow & & \sim \downarrow \\ b_1 & \longrightarrow & b_2 & \longrightarrow & \cdots & \longrightarrow & b_{n-1} & \longrightarrow & b_n \end{array}$$

with the vertical maps weak equivalences. We argue by induction on n , the base case of $n = 1$ following from the fact that ϕ is a DK-equivalence and all weak equivalences have homotopy inverses. Having constructed the diagram

$$\begin{array}{ccccccc} \phi(a_1) & \longrightarrow & \phi(a_2) & \longrightarrow & \cdots & \longrightarrow & \phi(a_{n-1}) \\ \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\ b_1 & \longrightarrow & b_2 & \longrightarrow & \cdots & \longrightarrow & b_{n-1} & \longrightarrow & b_n \end{array}$$

by induction, we know from [6, 1.4] that the homotopy category of objects in \mathcal{C} under a_{n-1} is equivalent to the homotopy category of objects in \mathcal{D} under $\phi(a_{n-1})$. We then get an object a' a map $a_{n-1} \rightarrow a'$ in \mathcal{C} and a zigzag of weak equivalences under $\phi(a_{n-1})$ in \mathcal{D} from $\phi(a')$ to b_n . Since b_n is in \mathcal{B} , by the embedding hypotheses, we see that a' is in \mathcal{A} . Using an appropriate generalized interval J , we let $a_n = (a_{n-1} \otimes J) \cup_{a_{n-1}} a'$. The inclusion of a_{n-1} in a_n is a cofibration in \mathcal{C} , and we get a weak equivalence under $\phi(a_{n-1})$ from $\phi(a_n)$ to b_n . To complete the argument we need to see that $a_{n-1} \rightarrow a_n$ is a cofibration in \mathcal{A} , i.e., that its cofiber is in \mathcal{A} . This follows since $\phi(a_n/a_{n-1})$ is weakly equivalent to b_n/b_{n-1} , which is in \mathcal{B} since by hypothesis $b_{n-1} \rightarrow b_n$ is a cofibration in \mathcal{B} . \square

Waldhausen constructed the K -theory spectrum $K\mathcal{C}_0$ as $w_\bullet S_\bullet^{(n)} \mathcal{C}_0$, where $S_\bullet^{(n)}$ is the iterated S_\bullet -construction and w_\bullet is the nerve of the subcategory of weak equivalences. The previous proposition extends the iterated S_\bullet construction to simplicially enriched categories. We could likewise consider the simplicially enriched categories $w_n \mathcal{C}$ with objects the sequences of weak equivalences

$$a_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} a_n$$

and simplicial sets of maps the natural transformations. Then for objects A and B , the simplicial set of $w_n \mathcal{C}(A, B)$ becomes

$$\mathcal{C}(a_0, b_0) \times_{\mathcal{C}(a_0, b_1)} \cdots \times_{\mathcal{C}(a_{n-1}, b_n)} \mathcal{C}(a_n, b_n).$$

While this works formally, it does not work well homotopically because the pullbacks are not over fibrations and so the mapping spaces are not homotopy limits.

We can sometimes resolve this problem by working with the simplicially enriched categories $\bar{w}_n\mathcal{C}$, where the objects are the sequences of maps which are weak equivalences and cofibrations; we use this construction in Section 8. However, this is often inconvenient and does not always produce the correct result, and so instead we describe a general technique for fixing the problem by putting choices of homotopies in the mapping spaces. As a first case, consider the following construction.

Construction 3.7. Let \mathcal{C} be a simplicially enriched category and let $v\mathcal{C}_0$ be a subcategory of \mathcal{C}_0 . We construct a topologically enriched category $v_1^M\mathcal{C}$ as follows. An object consists of a map $\alpha_0: a_0 \rightarrow a_1$ in $v\mathcal{C}_0$. The space of maps $v_1\mathcal{C}(A, B)$ consists of elements f_0, f_1 of the geometric realizations $|\mathcal{C}(a_0, b_0)|, |\mathcal{C}(a_1, b_1)|$ (respectively), a non-negative real number r , and a homotopy $f_{0,1}$ of length r in $|\mathcal{C}(a_0, b_1)|$ from $\beta_0 \circ f_0$ to $f_1 \circ \alpha$; we topologize this as a subspace of

$$|\mathcal{C}(a_0, b_0)| \times |\mathcal{C}(a_1, b_1)| \times \mathbb{R} \times |\mathcal{C}(a_0, b_1)|^I.$$

Composition is induced by composition of maps and homotopies.

In the notation “ M ” stands for Moore, as this employs the Moore trick for making homotopy composition associative. In this construction, the mapping space $v_1^M\mathcal{C}(A, B)$ is homotopy equivalent to the homotopy pullback

$$|\mathcal{C}(a_0, b_0)| \times_{|\mathcal{C}(a_0, b_1)|} |\mathcal{C}(a_0, b_1)|^I \times_{|\mathcal{C}(a_0, b_1)|} |\mathcal{C}(a_1, b_1)|.$$

The Moore trick generalizes from paths to maps out of higher simplices [19, §2]. We understand the n -simplex of length $r > 0$ to be the subspace $\Delta[n]_r$ of points (t_0, t_1, \dots, t_n) of \mathbb{R}^{n+1} with $0 \leq t_i \leq r$ and $\sum t_i = r$. Then given $r, s > 0$, the maps $\sigma_{r,s}^{i,n-i}: \Delta[i]_r \times \Delta[n-i]_s \rightarrow \Delta[n]_{r+s}$ defined by

$$\sigma_{r,s}^{i,n-i}: (t_0, \dots, t_i), (u_0, \dots, u_{n-i}) \mapsto (t_0, \dots, t_i + u_0, u_1, \dots, u_{n-i})$$

decompose $\Delta[n]_{r+s}$ as a union of prisms

$$\psi_{r,s}^n: \Delta[n]_{r+s} \cong \bigcup_{i=0}^n \Delta[i]_r \times \Delta[n-i]_s.$$

(See Proof of Theorem 2.4 in [19, p. 162].) This decomposition clearly commutes with the simplicial face and degeneracy operations, and it is associative in that the following diagram commutes.

$$\begin{array}{ccc} \Delta[i]_q \times \Delta[j]_r \times \Delta[k]_s & \xrightarrow{\sigma_{q,r}^{i,j} \times \text{id}} & \Delta[i+j]_{q+r} \times \Delta[k]_s \\ \text{id} \times \sigma_{r,s}^{j,k} \downarrow & & \downarrow \sigma_{q+r,s}^{i+j,k} \\ \Delta[i]_q \times \Delta[j+k]_{r+s} & \xrightarrow{\sigma_{q,r+s}^{i,j+k}} & \Delta[i+j+k]_{q+r+s} \end{array}$$

Construction 3.8 (Moore Nerve). For \mathcal{C} a simplicially enriched category and $v\mathcal{C}_0$ a subcategory of \mathcal{C}_0 , define the topologically enriched category $v_n^M\mathcal{C}$ as follows. The objects consist of the sequences of n composable maps in $v\mathcal{C}_0$

$$a_0 \xrightarrow{v} \dots \xrightarrow{v} a_n.$$

For convenience in what follows, we denote the structure map $a_i \rightarrow a_j$ as $\alpha_{i,j}$, for $i \leq j$ (and $\beta_{i,j}$, $\gamma_{i,j}$ similarly for objects B, C). An element of the space of maps from A to B consists of the following data:

- (i) An non-negative real number r
- (ii) For each $0 \leq m \leq n$ and each $0 \leq i_0 < \dots < i_m < n$ a map

$$f_{i_0, \dots, i_m} : \Delta[m]_r \longrightarrow |\mathcal{C}(a_{i_0}, b_{i_m})|$$

for $r > 0$, or an element of $|\mathcal{C}(a_{i_0}, b_{i_m})|$ for $r = 0$.

such that for any subset $i_{j_0}, \dots, i_{j_\ell}$ of i_1, \dots, i_m , the map

$$\beta_{i_{j_\ell}, i_m} \circ f_{i_{j_0}, \dots, i_{j_\ell}} \circ \alpha_{i_0, i_{j_0}} : \Delta[\ell]_r \longrightarrow |\mathcal{C}(a_{i_0}, b_{i_m})|$$

is the restriction to the face of f_{i_0, \dots, i_m} spanned by $i_{j_0}, \dots, i_{j_\ell}$. We topologize this as a subset of the evident product. Composition is induced by the prismatic decomposition above: for $F: A \rightarrow B$ of length $r > 0$ and $G: B \rightarrow C$ of length $s > 0$, the composition $H: A \rightarrow C$ of length $r + s$ is defined by taking h_{i_0, \dots, i_m} to be the map

$$(g_{i_j, \dots, i_m}(u_0, \dots, u_{m-j}) \circ \alpha_{i_0, i_j}) \circ (\gamma_{i_j, i_m} \circ f_{i_0, \dots, i_j}(t_0, \dots, t_j))$$

on the $\Delta[j]_r \times \Delta[m-j]_s$ prism in the $\psi_{r,s}^m$ decomposition of $\Delta[m]_{r+s}$. For $r = 0$ or $s = 0$, composition is induced by composition in \mathcal{C} .

A straightforward check of the formulas verifies that this defines a topological category. Moreover, $v_\bullet^M \mathcal{C}$ assembles into a simplicial topological category with the following naturality property. (It applies in particular to the important special case $\mathcal{C} = \mathcal{D}$ with $v\mathcal{C}_0 \subsetneq v\mathcal{D}_0$.)

Proposition 3.9. *Given simplicially enriched categories \mathcal{C} and \mathcal{D} , a simplicially enriched functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ that takes $v\mathcal{C}_0$ into $v\mathcal{D}_0$ induces a topologically enriched simplicial functor $v_\bullet^M \mathcal{C} \rightarrow v_\bullet^M \mathcal{D}$.*

For objects A and B , $v_n^M(A, B)$ is homotopy equivalent to the homotopy end of $\mathcal{C}(a_i, b_i)$ for $n > 0$, while $v_0^M \mathcal{C}(a, b) = |\mathcal{C}(a, b)| \times [0, \infty)$. In particular \mathcal{C} includes in $v_0^M \mathcal{C}$ (after geometric realization) as the subcategory of maps of length zero. More generally, the nerve categories $v_n \mathcal{C}$ include (after geometric realization) as the subcategories of the Moore nerve categories $v_n^M \mathcal{C}$ of the maps of length zero. Restricting to simplicially enriched Waldhausen categories, we get the following proposition.

Proposition 3.10. *Let \mathcal{C} be a simplicially enriched Waldhausen category and $v\mathcal{C}_0$ a subcategory of \mathcal{C}_0 .*

- (i) *If $v\mathcal{C}_0 \subset w\mathcal{C}_0$, then the inclusion of \mathcal{C} in $v_n^M \mathcal{C}$ is a DK-equivalence.*
- (ii) *If $v\mathcal{C}_0 \subset \text{co } \mathcal{C}_0$, then the inclusion of $v_n \mathcal{C}$ in $v_n^M \mathcal{C}$ is a DK-equivalence.*

Finally, we use the following notation.

Definition 3.11. Let \mathcal{C} be a simplicial enriched Waldhausen category, and let $v\mathcal{C}_0$ be a subcategory of \mathcal{C}_0 . Define $v_\bullet^M \mathcal{C}^\Gamma$ to be the simplicial spectral category obtained from the simplicial Γ -category with

$$v_n^M \mathcal{C}_q^\Gamma(X, Y) = v_n^M \mathcal{C}(X, \bigvee_q Y).$$

For $\mathcal{A} \subset \mathcal{C}$ an enhanced simplicially enriched Waldhausen category, define $v_{\bullet}^M \mathcal{A}^S$ to be the simplicial spectral category with

$$v_n^M \mathcal{A}^S(X, Y)(q) = v_n^M \mathcal{C}(X, \Sigma^q Y).$$

In the formula, \bigvee denotes the entry-wise coproduct; although this is not the coproduct in $v_n^M \mathcal{C}$, we can identify

$$v_n^M \mathcal{C}(\bigvee_q Y, Z) \subset \prod_q v_n^M \mathcal{C}(Y, Z)$$

as the subspace of q -tuples of maps, all having the same length. We then obtain Γ -category composition as in Section 2. Likewise, in the enhanced context, although $\Sigma^n Y$ is not a based tensor in $v_n^M \mathcal{C}$, we nevertheless have a continuous functor

$$v_n^M \mathcal{C}(Y, Z) \longrightarrow v_n^M(\Sigma^n Y, \Sigma^n Z)$$

and we obtain the spectral category composition as in Section 2.

Finally, note that the resulting spectral categories are enriched in symmetric spectra of topological spaces, as opposed to simplicial sets. In the context of the definition of THH we study in Section 5, this distinction will not cause any difficulties.

4. THE MOORE S'_{\bullet} CONSTRUCTION

Although the S_{\bullet} construction translates naturally to the enriched context, it is often useful to be able to weaken the cocartesian condition in the construction and instead work with an equivalent construction defined in terms of *homotopy cocartesian* squares called the S'_{\bullet} construction [5, §2]. This flexibility plays a key role in the proof of the dévissage theorem for the localization theorem for $THH(ku)$ in Section 12. Such a definition also provides models of K -theory and THH which are functorial in functors “exact up to homotopy” as explained in Section 13. In this section we introduce an appropriately enriched version of the S'_{\bullet} construction, using the Moore ideas from the previous section to construct the homotopically correct enrichment.

We begin by reviewing the S'_{\bullet} construction. For this, recall from [6, §2] that a *weak cofibration* is a map that is weakly equivalent (by a zigzag) to a cofibration in the category $\text{Ar} \mathcal{C}_0$ of arrows in \mathcal{C}_0 , and a *homotopy cocartesian square* is a square diagram that is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of cofibrations.

Construction 4.1. Let \mathcal{C}_0 be a Waldhausen category. Define $S'_n \mathcal{C}_0$ to be the full subcategory of functors $A: \text{Ar}[n] \rightarrow \mathcal{C}_0$ such that:

- The initial map $* \rightarrow a_{i,i}$ is a weak equivalence for all i ,
- The map $a_{i,j} \rightarrow a_{i,k}$ is a weak cofibration for all $i \leq j \leq k$, and
- The diagram

$$\begin{array}{ccc} a_{i,j} & \longrightarrow & a_{i,k} \\ \downarrow & & \downarrow \\ a_{j,j} & \longrightarrow & a_{j,k} \end{array}$$

is a homotopy cocartesian square for all $i \leq j \leq k$.

We define a map $A \rightarrow B$ to be a weak equivalence when each $a_{i,j} \rightarrow b_{i,j}$ is a weak equivalence. Clearly S'_{\bullet} assembles into a simplicial category with the usual face and degeneracy functors.

As in the previous section, we obtain a connective spectral enrichment using the objectwise coproduct and (when defined) a non-connective spectral enrichment using the objectwise suspension.

We use analogous notation for the enriched categories associated to full subcategories of diagram categories, obtaining for example $S_n^M \mathcal{C}$ and $S_n'^M \mathcal{C}$ as full subcategories of $\mathrm{Ar}[n] \mathcal{C}$. Because the Moore Tot always has the homotopy type of the homotopy end (containing it as a deformation retract), we obtain the following result as an immediate consequence.

Proposition 4.5. *For a simplicially enriched Waldhausen category \mathcal{C} , the inclusion of the topologically enriched category $|S_n \mathcal{C}|$ in $S_n^M \mathcal{C}$ as the length zero part is a DK-equivalence.*

Considering more complicated diagrams, this also applies to $w_p S_{q_1, \dots, q_n}^{(n)} \mathcal{C}$. Thinking of these categories as subcategories of $w_p S_{q_1, \dots, q_n}'^{(n)} \mathcal{C}$, the more restricted homotopies in $w_p^M S_{q_1, \dots, q_n}^{(n)} \mathcal{C}$ make its mapping spaces subspaces of $(w_p S_{q_1, \dots, q_n}'^{(n)})^M \mathcal{C}$, and we get the following result.

Proposition 4.6. *For a simplicially enriched Waldhausen category \mathcal{C} , the inclusion of $w_p^M S_{q_1, \dots, q_n}^{(n)} \mathcal{C}$ in $(w_p S_{q_1, \dots, q_n}'^{(n)})^M \mathcal{C}$ is a DK-embedding. If \mathcal{C}_0 admits FFWC, then it is a DK-equivalence.*

We write $(w_p S_{q_1, \dots, q_n}'^{(n)})^M \mathcal{C}^\Gamma$ and when appropriate $(w_p S_{q_1, \dots, q_n}'^{(n)})^M \mathcal{C}^S$ for the associated spectrally enriched categories, now enriched in topological symmetric spectra.

5. THH , TC , AND THE CYCLOTOMIC TRACE

In this section, we apply the constructions of THH and TC of spectral categories [4, §3] in the context of the spectral enrichments associated to a simplicially enriched Waldhausen category \mathcal{C} . For the connective enrichments, we require Waldhausen's S_\bullet construction in order to properly handle extension sequences in the Waldhausen structure for reasons first observed by McCarthy [18, 3.3.5] and Dundas-McCarthy [8, 2.3.4]; for the non-connective enrichment, the S_\bullet construction turns out to be superfluous.

We understand the construction of THH of a spectral category in terms of the Bökstedt-Dundas-McCarthy model of the Hochschild-Mitchell “cyclic nerve”. Its fundamental properties include:

- (i) THH is a continuous functor from spectral categories (in topological symmetric spectra) to cyclotomic spectra.
- (ii) THH sends DK-equivalences of spectral categories to weak equivalences of cyclotomic spectra.

Associated to the cyclotomic spectrum THH are natural pro-spectra TR and TC . Their fundamental properties include:

- (i) If a spectral functor induces a weak equivalence on THH , then it induces a level weak equivalence of pro-spectra on TR and TC .
- (ii) The TR and TC constructions preserve cofiber sequences of cyclotomic spectra.

For details and constructions, we refer the reader to [4, §3] and the references therein. The construction makes sense for spectral categories enriched in symmetric

spectra in either simplicial sets or topological spaces; in the case of simplicial sets, essentially the first step is geometric realization. The treatment in [4] focuses on the context of simplicial sets; the properties listed there still hold in the context of topological spaces with the additional technical hypothesis that the individual mapping spectra in the spectral categories be non-degenerately based (the inclusion of the base point in each space of the symmetric spectrum is a cofibration) and that the inclusion of the identity elements (in zeroth spaces) are cofibrations. This happens automatically for the spectral categories obtained by geometric realization from spectral categories in the simplicial set context and more generally for all of the spectral categories we work with in this paper.

Because the properties of TR and TC are so closely tied to the properties of THH , in later sections we will typically state weak equivalence and cofiber sequence results in terms of THH , as these then automatically imply the corresponding results for TR and TC .

Definition 5.1. For a simplicially enriched Waldhausen category \mathcal{C} , we define

$$\begin{aligned} WTHH^\Gamma \mathcal{C} &= \Omega |THH(S_\bullet \mathcal{C}^\Gamma)| \\ WTR^\Gamma \mathcal{C} &= \Omega |TR(S_\bullet \mathcal{C}^\Gamma)| \\ WTC^\Gamma \mathcal{C} &= \Omega |TC(S_\bullet \mathcal{C}^\Gamma)|. \end{aligned}$$

If \mathcal{C} is a simplicially tensored Waldhausen category and $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicial Waldhausen category, then we define

$$\begin{aligned} WTHH \mathcal{A} &= \Omega |THH(S_\bullet \mathcal{A}^S)| \\ WTR \mathcal{A} &= \Omega |TR(S_\bullet \mathcal{A}^S)| \\ WTC \mathcal{A} &= \Omega |TC(S_\bullet \mathcal{A}^S)|. \end{aligned}$$

In other words, we apply THH , TR , or TC first to get simplicial (or multisimplicial) cyclotomic spectra or pro-spectra. Then we take the geometric realization in the simplicial directions, followed by loops. We have the following naturality properties.

Proposition 5.2. *An enriched exact functor induces maps on $WTHH^\Gamma$, WTR^Γ , and WTC^Γ . A tensored exact or enhanced exact functor induces maps on $WTHH$, WTR , and WTC . Naturally weakly equivalent functors induce the same map in the stable category.*

Proof. The only part not immediate from the construction is the last statement. We use Construction 3.7 for $v\mathcal{D}_0$ the subcategory of weak equivalences $w\mathcal{D}_0$. We have a pair of simplicial spectrally enriched functors $w_1^M S_\bullet \mathcal{D}^\Gamma \rightarrow S_\bullet \mathcal{D}^\Gamma$ each split by the inclusion $S_\bullet \mathcal{D}^\Gamma \rightarrow w_1^M S_\bullet \mathcal{D}^\Gamma$. Since the inclusion induces a DK-equivalence $S_\bullet \mathcal{D}^\Gamma \rightarrow w_1^M S_\bullet \mathcal{D}^\Gamma$, both maps $w_1^M S_\bullet \mathcal{D}^\Gamma \rightarrow S_\bullet \mathcal{D}^\Gamma$ induce the same map in the stable category on THH , TR , and TC . Now, given enriched exact functors $\phi_0, \phi_1 : \mathcal{C} \rightarrow \mathcal{D}$ and h a natural weak equivalence between them, we get a simplicial spectrally enriched functor $S_\bullet \mathcal{C}^\Gamma \rightarrow w_1^M S_\bullet \mathcal{D}^\Gamma$ (factoring through the length zero part $w_1 S_\bullet \mathcal{D}^\Gamma$). The two composites

$$S_\bullet \mathcal{C}^\Gamma \longrightarrow w_1^M S_\bullet \mathcal{D}^\Gamma \longrightarrow S_\bullet \mathcal{D}^\Gamma$$

are the maps induced by ϕ_0 and ϕ_1 . For tensored exact or enhanced exact functors ϕ_0 and ϕ_1 and a natural weak equivalence between them, the same argument applies

to show that the maps on $WTHH$, WTR , and WTC induced by ϕ_0 and ϕ_1 coincide in the stable category. \square

Applying Proposition 3.4 and 3.6 we obtain the following homotopy invariance properties.

Proposition 5.3. *Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be an enriched exact functor between simplicially enriched Waldhausen categories that are DK-compatible. Assume that ϕ is a DK-embedding and that every object of $S_n \mathcal{D}$ is weakly equivalent to an object in the image of $S_n \phi$ (for all n). Then ϕ induces weak equivalences on $WTHH^\Gamma$, WTR^Γ , and WTC^Γ .*

Proposition 5.4. *Let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories. If ϕ is a DK-embedding of the ambient simplicially tensored categories and a DK-equivalence $\mathcal{A} \rightarrow \mathcal{B}$ and a DK-equivalence on the ambient simplicially tensored Waldhausen categories, then ϕ induces a weak equivalence on $WTHH$, WTR , and WTC .*

Implicitly in the previous propositions we passed from a level weak equivalence of simplicial spectra $X_\bullet \rightarrow Y_\bullet$ to a weak equivalence on geometric realization $|X_\bullet| \rightarrow |Y_\bullet|$. Using the standard geometric realization, we need hypotheses on X_\bullet and Y_\bullet for this to work. One sufficient hypothesis is that X_\bullet and Y_\bullet are *spacewise proper*: we say that a simplicial symmetric spectrum of topological spaces X_\bullet is space-wise proper when the simplicial space $X_\bullet(n)$ is proper for every n , i.e., for each k , each degeneracy map $X_k(n) \rightarrow X_{k+1}(n)$ is a Hurewicz cofibration (satisfies the homotopy extension property). The following proposition applies to verify this property for the constructions in the previous propositions and the many other constructions in this paper. Its proof requires the details of the THH construction but is then straightforward given the standard properties of Hurewicz cofibrations.

Proposition 5.5. *Let \mathcal{C}_\bullet be a simplicial object in the category of spectral categories (in topological symmetric spectra). Assume that for all k and all objects x, y of \mathcal{C}_k , each space of the mapping spectrum $\mathcal{C}_k(x, y)(n)$ is non-degenerately based and each degeneracy map $s^i: \mathcal{C}_k(x, y)(n) \rightarrow \mathcal{C}_{k+1}(s^i x, s^i y)(n)$ is a Hurewicz cofibration. Then the simplicial spectrum $THH(\mathcal{C}_\bullet)$ is spacewise proper.*

Waldhausen's approximation property provides a convenient formulation for the conditions in Propositions 5.3 and 5.4 that often holds in practice. We say that exact functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$ has the *approximation property* when:

- (i) A map $f: a \rightarrow b$ is a weak equivalence in \mathcal{C} only if the map $\phi(f)$ in \mathcal{D} is a weak equivalence.
- (ii) For every map $f: \phi(a) \rightarrow x$ in \mathcal{D} , there exists a map $g: a \rightarrow b$ in \mathcal{C} and a weak equivalence $h: \phi(b) \rightarrow x$ in \mathcal{D} such that $g = h \circ \phi(g)$.

We then have the following THH analogue of Waldhausen's Approximation Theorem. The proof is that under factorization hypotheses, the approximation property implies that ϕ is a DK-equivalence; see [6, 1.4–1.5].

Theorem 5.6. *Let $\phi: (\mathcal{A} \subset \mathcal{C}) \rightarrow (\mathcal{B} \subset \mathcal{D})$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories, and suppose that $\phi_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ satisfies the approximation property. If every object of \mathcal{B} is weakly equivalent to the image of an object of \mathcal{A} , then ϕ induces weak equivalences on $WTHH^\Gamma$, WTR^Γ , WTC^Γ and on $WTHH$, WTR , WTC .*

In many situations, the underlying Waldhausen category \mathcal{C}_0 of a simplicially enriched Waldhausen category \mathcal{C} admits a second subcategory of weak equivalences $v\mathcal{C}_0$ (not necessarily related to the simplicial structure, or even satisfying the two out of three property). When $v\mathcal{C}_0$ contains all the isomorphisms and satisfies the Gluing Axiom (Weq 2 in [28, §1.2]), each Waldhausen category $S_n\mathcal{C}_0$ inherits a subcategory $vS_n\mathcal{C}_0$ also satisfying these properties. In this context, we have additional variants of THH , TR , and TC .

Definition 5.7. Let \mathcal{C} be a simplicially enriched Waldhausen category, and let $v\mathcal{C}_0$ be a subcategory of \mathcal{C}_0 containing all the isomorphisms and satisfying the Gluing Axiom. Then we define the connective relative THH , TC , and TR of $(\mathcal{C}|v)$ as indicated below (on the left). When \mathcal{C} is simplicially tensored and $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicially enriched Waldhausen category, we define the non-connective THH , TC , and TR of $(\mathcal{A}|v)$ as indicated below (on the right).

$$\begin{aligned} WTHH^\Gamma(\mathcal{C}|v) &= \Omega|THH(v_\bullet^M S_\bullet \mathcal{C}^\Gamma)| & WTHH(\mathcal{A}|v) &= \Omega|THH(v_\bullet^M S_\bullet \mathcal{A}^S)| \\ WTR^\Gamma(\mathcal{C}|v) &= \Omega|TR(v_\bullet^M S_\bullet \mathcal{C}^\Gamma)| & WTR(\mathcal{A}|v) &= \Omega|TR(v_\bullet^M S_\bullet \mathcal{A}^S)| \\ WTC^\Gamma(\mathcal{C}|v) &= \Omega|TC(v_\bullet^M S_\bullet \mathcal{C}^\Gamma)| & WTC(\mathcal{A}|v) &= \Omega|TC(v_\bullet^M S_\bullet \mathcal{A}^S)| \end{aligned}$$

In the special case when $v\mathcal{C}_0$ is the category of weak equivalences $w\mathcal{C}_0$, the inclusion of each $S_n\mathcal{C}^\Gamma$ into $w_q^M S_n\mathcal{C}^\Gamma$ and (when defined) $S_n\mathcal{A}^S$ into $w_q^M S_n\mathcal{A}^S$ is a DK-equivalence. This implies the following proposition.

Proposition 5.8. For \mathcal{C} a simplicially enriched Waldhausen category and \mathcal{A} an enhanced simplicial enriched Waldhausen category, the maps

$$\begin{aligned} WTHH^\Gamma \mathcal{C} &\longrightarrow WTHH^\Gamma(\mathcal{C}|w) & WTHH \mathcal{A} &\longrightarrow WTHH(\mathcal{A}|w) \\ WTR^\Gamma \mathcal{C} &\longrightarrow WTR^\Gamma(\mathcal{C}|w) & WTR \mathcal{A} &\longrightarrow WTR(\mathcal{A}|w) \\ WTC^\Gamma \mathcal{C} &\longrightarrow WTC^\Gamma(\mathcal{C}|w) & WTC \mathcal{A} &\longrightarrow WTC(\mathcal{A}|w) \end{aligned}$$

are weak equivalences

To construct the cyclotomic trace, we need a final variant of these constructions where we iterate the S_\bullet construction.

Definition 5.9. Let

$$\widetilde{WTHH}^\Gamma \mathcal{C}(n) = |THH(w_\bullet^M S_\bullet^{(n)} \mathcal{C}^\Gamma)|.$$

The simplicial maps of spectral categories

$$\Sigma_\bullet^{n-m} w_\bullet^M S_\bullet^{(m)} \mathcal{C}^\Gamma \longrightarrow w_\bullet^M S_\bullet^{(n)} \mathcal{C}^\Gamma$$

induce maps

$$\Sigma^{n-m} \widetilde{WTHH}^\Gamma \mathcal{C}(m) \longrightarrow \widetilde{WTHH}^\Gamma \mathcal{C}(n)$$

(as in [28, §1.3]). These maps assemble $\widetilde{WTHH}^\Gamma \mathcal{C}$ into a symmetric spectrum in the category of cyclotomic spectra. We define \widetilde{WTR} and \widetilde{WTC} to be the TR and TC pro-spectra constructed from \widetilde{WTHH} .

As a consequence of the Additivity Theorem 6.1, we prove the following lemma in Section 6.

Lemma 5.10. The map $\Sigma^{n-m} \widetilde{WTHH}^\Gamma \mathcal{C}(m) \rightarrow \widetilde{WTHH}^\Gamma \mathcal{C}(n)$ in Definition 5.9 is a weak equivalence for all $n \geq m > 0$.

We have analogous constructions and results in the relative case (using v_{\bullet}^M in place of w_{\bullet}^M) and non-connective case (using \mathcal{A}^S for \mathcal{A}^{Γ} when \mathcal{A} is enhanced). The identity $WTHH^{\Gamma}\mathcal{C} = \Omega\widetilde{WTHH}^{\Gamma}\mathcal{C}(1)$ then immediately implies the following result.

Theorem 5.11. *We have natural isomorphisms in the stable category*

$$WTHH^{\Gamma}\mathcal{C} \simeq \widetilde{WTHH}^{\Gamma}\mathcal{C} \quad WTR^{\Gamma}\mathcal{C} \simeq \widetilde{WTR}^{\Gamma}\mathcal{C} \quad WTR^{\Gamma}\mathcal{C} \simeq \widetilde{WTC}^{\Gamma}\mathcal{C}$$

and likewise for the relative and non-connective variants when these are defined.

We can now define the cyclotomic trace.

Definition 5.12. For a simplicially enriched Waldhausen category \mathcal{C} , the cyclotomic trace

$$K(\mathcal{C}_0) \longrightarrow \widetilde{WTC}^{\Gamma}\mathcal{C} \longrightarrow \widetilde{WTHH}^{\Gamma}\mathcal{C}$$

is the map induced by the inclusion of objects

$$K\mathcal{C}_0 = \text{Ob}(w_{\bullet}S_{\bullet}^{(n)}\mathcal{C}_0) = \text{Ob}(w_{\bullet}^M S_{\bullet}^{(n)}\mathcal{C}^{\Gamma}) \longrightarrow \widetilde{WTHH}^{\Gamma}\mathcal{C}.$$

For $v\mathcal{C}_0$ a subcategory of \mathcal{C}_0 containing the isomorphisms and satisfying the Gluing Axiom, the relative cyclotomic trace is the map

$$K(\mathcal{C}_0|v) \longrightarrow \widetilde{WTC}^{\Gamma}(\mathcal{C}|v) \longrightarrow \widetilde{WTHH}^{\Gamma}(\mathcal{C}|v)$$

induced by the inclusion of objects

$$K(\mathcal{C}_0|v) = \text{Ob}(v_{\bullet}S_{\bullet}^{(n)}\mathcal{C}_0) = \text{Ob}(v_{\bullet}^M S_{\bullet}^{(n)}\mathcal{C}^{\Gamma}) \longrightarrow \widetilde{WTHH}^{\Gamma}(\mathcal{C}|v).$$

Finally, to compare the definitions of this section with the theories used in [4], we state the following two theorems. The first is a consequence of the Additivity Theorem 6.3 and proved in Section 6.

Theorem 5.13. *Let \mathcal{A} be an enhanced simplicial Waldhausen category. The inclusion of $THH(\mathcal{A}^S)$ in $WTHH(\mathcal{A})$ is a weak equivalence of cyclotomic spectra.*

The second is a special case of the Sphere Theorem from Section 9; see Corollaries 9.4 and 9.12.

Theorem 5.14. *Let R be a ring, a simplicial ring, or a connective ring spectrum, and let \mathcal{A} be the simplicially tensored Waldhausen category of finite cell modules (built out of free or finitely generated projective modules) in Example 1.5.(i) or (ii) (as appropriate). Then the natural map $WTHH^{\Gamma}(\mathcal{A}) \rightarrow WTHH(\mathcal{A})$ is a weak equivalence of cyclotomic spectra.*

Thus, for a ring, simplicial ring, or connective ring spectrum, we have weak equivalences of cyclotomic spectra

$$WTHH^{\Gamma}(\mathcal{A}) \xrightarrow{\sim} WTHH(\mathcal{A}) \xleftarrow{\sim} THH(\mathcal{A}) \xleftarrow{\sim} THH(R),$$

where the last weak equivalence is a special case of [4, 4.12].

Part 2. K -theory theorems in THH and TC

6. THE ADDITIVITY THEOREM

In this section, we present the Additivity Theorem for the THH of Waldhausen categories. The modern viewpoint, implicit in [28] but first written explicitly by Staffeldt [25], holds the Additivity Theorem as the fundamental property of K -theory. Following this perspective, we deduce the remaining K -theoretic properties of THH from the Additivity Theorem in the next three sections.

To state the Additivity Theorem, we use the following notation. For \mathcal{C} a simplicially enriched Waldhausen category, let $\mathcal{E}(\mathcal{C}) = S_2\mathcal{C}$ be the simplicially enriched Waldhausen category with objects the cofiber sequences $x \rightarrow y \rightarrow z$ (in \mathcal{C}_0). We have enriched exact functors α, β, γ from $\mathcal{E}(\mathcal{C})$ to \mathcal{C} defined by

$$\alpha(x \rightarrow y \rightarrow z) = x, \quad \beta(x \rightarrow y \rightarrow z) = y, \quad \gamma(x \rightarrow y \rightarrow z) = z.$$

Theorem 6.1 (Additivity Theorem). *For a simplicially enriched Waldhausen category \mathcal{C} , the enriched exact functors α and γ induce a weak equivalence of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{E}(\mathcal{C})) \longrightarrow WTHH^\Gamma(\mathcal{C}) \times WTHH^\Gamma(\mathcal{C}) \simeq WTHH^\Gamma(\mathcal{C}) \vee WTHH^\Gamma(\mathcal{C}).$$

McCarthy's proof of the Additivity Theorem for K -theory [17] provides a very general argument for showing that the map $(\alpha, \gamma): S_\bullet\mathcal{E}(\mathcal{C}) \rightarrow S_\bullet\mathcal{C} \times S_\bullet\mathcal{C}$ induces a homotopy equivalence in various contexts. The elaboration in [18, §3.4-3.5] to prove the Additivity Theorem for cyclic homology of k -linear categories carries over essentially word for word to prove the Additivity Theorem above, just replacing “ CN ” with “ THH ” and “ k -linear” with “spectral”. (The only property of THH or CN needed is that it takes simplicial homotopy equivalences of simplicial (spectrally or k -linearly) enriched categories to weak equivalences of spectra or simplicial sets.)

The following result is both a generalization and a corollary of the Additivity Theorem above. Recall that a sequence of natural transformations of exact functors $f \rightarrow g \rightarrow h$ from \mathcal{C}_0 to \mathcal{D}_0 forms a *cofiber sequence of exact functors*, when (taken together) they define an exact functor \mathcal{C} to $\mathcal{E}(\mathcal{D})$.

Corollary 6.2. *Let \mathcal{C} and \mathcal{D} be simplicially enriched Waldhausen categories, and let $f \rightarrow g \rightarrow h$ be a sequence of enriched exact functors $\mathcal{C} \rightarrow \mathcal{D}$ that forms a cofiber sequence of exact functors. Then the maps*

$$WTHH^\Gamma(g) \quad \text{and} \quad WTHH^\Gamma(f) \vee WTHH^\Gamma(h).$$

from $WTHH^\Gamma(\mathcal{C})$ to $WTHH^\Gamma(\mathcal{D})$ agree in the homotopy category of cyclotomic spectra.

Proof. The functor $\mathcal{D} \times \mathcal{D} \rightarrow \mathcal{E}(\mathcal{D})$ sending (a, b) to $a \rightarrow a \vee b \rightarrow b$ is a enriched exact functor, and the composite map

$$WTHH^\Gamma(\mathcal{D}) \vee WTHH^\Gamma(\mathcal{D}) \longrightarrow WTHH^\Gamma(\mathcal{D} \times \mathcal{D}) \longrightarrow WTHH^\Gamma(\mathcal{E}(\mathcal{D}))$$

splits the zigzag of weak equivalences in the Additivity Theorem and is therefore a weak equivalence of cyclotomic spectra. It follows that β and $\alpha \vee \gamma$ induce the same map $\mathcal{E}(\mathcal{D}) \rightarrow \mathcal{D}$ in the homotopy category of cyclotomic spectra. Precomposing with $f \rightarrow g \rightarrow h$ proves the corollary. \square

This corollary provides the key tool for even more general additivity statements. For example, the map $S_n\mathcal{C} \rightarrow \mathcal{C} \times S_{n-1}\mathcal{C}$ defined by sending $X = (x_{i,j})$ to $(x_{0,1}, d_0X)$

induces a weak equivalence on $WTHH^\Gamma$. To see this, consider the map $\mathcal{C} \times S_{n-1}\mathcal{C} \rightarrow S_n\mathcal{C}$ sending (x, Y) to $Z = (z_{i,j})$ with

$$z_{i,j} = \begin{cases} x \vee y_{0,j-1} & i = 0 \\ y_{i-1,j-1} & i > 0. \end{cases}$$

The composite map on $\mathcal{C} \times S_{n-1}\mathcal{C}$ is the identity, and the composite map on $S_n\mathcal{C}$ is $f \vee h$ for exact functors f and h that fit in a cofiber sequence of exact functors $f \rightarrow g \rightarrow h$ with g the identity. We will use this argument many times in what follows.

When \mathcal{A} is a simplicially enhanced Waldhausen category, so is $\mathcal{E}(\mathcal{A})$ and the functors α, β, γ are enhanced exact. We have precise analogues of the previous results (with the same proof). In fact, we have the following stronger version of the Additivity Theorem for the non-connective enrichment as a consequence of the analogue of the Thomason-Trobaugh localization theorem for THH [4, 6.1] (q.v. op. cit. B.8).

Theorem 6.3. *Let \mathcal{A} be an enhanced simplicial Waldhausen category. The enhanced exact functors α and γ induce a weak equivalence of cyclotomic spectra*

$$THH(\mathcal{E}^S(\mathcal{A})) \longrightarrow THH(\mathcal{A}^S) \times THH(\mathcal{A}^S) \longleftarrow THH(\mathcal{A}^S) \vee THH(\mathcal{A}^S).$$

As a consequence of the Additivity Theorems 6.1 and 6.3, we can now prove Lemma 5.10 and Theorem 5.13.

Proof of Lemma 5.10 and Theorem 5.13. We will treat Lemma 5.10 in detail; Theorem 5.13 follows from the same argument using \mathcal{A} in place of $S_\bullet\mathcal{C}$ and Theorem 6.3 in place of Theorem 6.1. To prove Lemma 5.10, it suffices to show that the map $\Sigma WTHH^\Gamma(\mathcal{C}) \rightarrow WTHH^\Gamma(S_\bullet\mathcal{C})$ is a weak equivalence. This map is induced by a simplicial map

$$\bigvee_n WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(S_n\mathcal{C}),$$

which is a weak equivalence by the Additivity Theorem and the argument following Corollary 6.2. \square

7. THE COFIBER THEOREM

This section is the first of three that apply the Additivity Theorem to prove standard K -theory theorems in THH and TC . This section provides a general cofibration sequence for THH (and TC) associated to a map of Waldhausen categories by identifying the cofiber term as a version of THH . Following Waldhausen [28, §1.5], we call this theorem the “Cofiber Theorem”.

We begin with the construction of the cofiber term. For $f: \mathcal{C} \rightarrow \mathcal{D}$ an enriched exact functor, we define a simplicially enriched Waldhausen category $S_n f$ as follows. An object consists of an object $Y = (y_{i,j})$ of $S_n\mathcal{C}$ together with an object $X = (x_{i,j})$ of $S_{n+1}\mathcal{D}$ such that $d_0 X = f(Y)$, that is, $x_{i+1,j+1} = f(y_{i,j})$, with the structure maps for this subdiagram in X identical with $f(Y)$. For objects $(X, Y), (X', Y')$, the simplicial set of maps consists of the simplicial set of natural transformations. We make this a Waldhausen category by declaring a map $(X, Y) \rightarrow (X', Y')$ to be a cofibration (resp., weak equivalence) when the restrictions $X' \rightarrow X$ (in $S_{n+1}\mathcal{D}$) and $Y' \rightarrow Y$ (in $S_n\mathcal{C}$) are both cofibrations (resp., weak equivalences). This assembles into a simplicial object in the category of simplicially enriched Waldhausen

categories using the usual face and degeneracy maps on $S_n\mathcal{C}$ and the last $n+1$ face and degeneracy maps on $S_{n+1}\mathcal{D}$.

Definition 7.1. For $f: \mathcal{C} \rightarrow \mathcal{D}$ an enriched exact functor, define

$$WTHH^\Gamma(f) = |WTHH^\Gamma(S_\bullet f)|.$$

We note that when $f: \mathcal{A} \rightarrow \mathcal{B}$ is an enhanced exact functor between enhanced simplicially enriched Waldhausen categories, then $S_n f$ is also an enhanced simplicially enriched Waldhausen category and $S_\bullet f$ is a simplicial object in enhanced simplicially enriched Waldhausen categories. We write $WTHH(f) = |WTHH(S_\bullet f)|$.

To put this construction in perspective, we have an alternative description of $S_\bullet f$ as a pullback. For any simplicial object Z_\bullet , we can form the “path” object PZ_\bullet precomposing with the shift operation $[n] \mapsto [n+1]$ in the category of standard simplices (or finite ordered sets). In this notation, we have a pullback square

$$\begin{array}{ccc} S_\bullet f & \xrightarrow{X} & PS_\bullet \mathcal{D} \\ Y \downarrow & & \downarrow d_0 \\ S_\bullet \mathcal{C} & \xrightarrow{f} & S_\bullet \mathcal{D} \end{array}$$

in the category of simplicial simplicially enriched categories. The usual extra degeneracy argument produces a simplicial null homotopy on $PS_\bullet \mathcal{C}$, and applying $WTHH^\Gamma$ and (when appropriate) $WTHH$, we get commutative squares of cyclotomic spectra

$$\begin{array}{ccc} WTHH^\Gamma(f) \longrightarrow |WTHH^\Gamma(PS_\bullet \mathcal{D})| & & WTHH(f) \longrightarrow |WTHH(PS_\bullet \mathcal{B})| \\ \downarrow & & \downarrow \\ |WTHH^\Gamma(S_\bullet \mathcal{C})| \longrightarrow |WTHH^\Gamma(S_\bullet \mathcal{D})| & & |WTHH(S_\bullet \mathcal{A})| \longrightarrow |WTHH(S_\bullet \mathcal{B})| \end{array}$$

where the top right entry comes with a canonical null homotopy through cyclotomic maps. We therefore get a map of cyclotomic spectra from $WTHH^\Gamma(f)$ to the homotopy fiber of the map $|WTHH^\Gamma(S_\bullet \mathcal{C})| \rightarrow |WTHH^\Gamma(S_\bullet \mathcal{D})|$, which is equivalent to the homotopy cofiber of the map $WTHH^\Gamma \mathcal{C} \rightarrow WTHH^\Gamma \mathcal{D}$. Likewise in the enhanced exact context, we get a map of cyclotomic spectra from $WTHH(f)$ to the homotopy cofiber of the map $WTHH \mathcal{A} \rightarrow WTHH \mathcal{B}$. The Cofiber Theorem asserts that these maps are weak equivalences.

Theorem 7.2 (Cofiber Theorem). *For $f: \mathcal{C} \rightarrow \mathcal{D}$ an enriched exact functor, we have a cofiber sequence of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\mathcal{D}) \longrightarrow WTHH^\Gamma(f) \longrightarrow |WTHH^\Gamma(S_\bullet \mathcal{C})|.$$

For $f: \mathcal{A} \rightarrow \mathcal{B}$ an enhanced exact functor, we have a cofiber sequence of cyclotomic spectra

$$WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{B}) \longrightarrow WTHH(f) \longrightarrow |WTHH(S_\bullet \mathcal{A})|.$$

Proof. (cf. [28, 1.5.5]) The argument for the connective and non-connective enrichments are identical; we treat the connective case in detail. Consider the map

$$WTHH^\Gamma(\mathcal{D}) \vee \bigvee_n WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\mathcal{D} \times S_n \mathcal{C}) \longrightarrow WTHH^\Gamma(S_n f)$$

induced by sending b, a_1, \dots, a_n to (b, Y) and then (X, Y) with $Y = (y_{i,j})$ for

$$y_{i,j} = a_{i+1} \vee \dots \vee a_j$$

and $X = (x_{i,j})$ for

$$x_{i,j} = \begin{cases} b \vee f(y_{0,j-1}) & i = 0 \\ f(y_{i-1,j-1}) & i > 0 \end{cases}$$

with the canonical maps induced by inclusions and quotients of summands. Applying the argument following Corollary 6.2, we see that this map is a weak equivalence. Letting n vary, these assemble into a simplicial map where we regard the domain as the simplicial cyclotomic spectrum

$$WTHH^\Gamma(\mathcal{D}) \cup_{WTHH^\Gamma(\mathcal{C})} WTHH^\Gamma(\mathcal{C}) \wedge \Delta[1].$$

On geometric realization, this induces a map from the homotopy cofiber

$$C = WTHH^\Gamma(\mathcal{D}) \cup_{WTHH^\Gamma(\mathcal{C})} WTHH^\Gamma(\mathcal{C}) \wedge I$$

to $WTHH^\Gamma(f)$ that we see is a weak equivalence. The composite map

$$C \longrightarrow WTHH^\Gamma(f) \longrightarrow WTHH^\Gamma(S_\bullet \mathcal{C})$$

factors as the connecting map $C \rightarrow \Sigma WTHH^\Gamma(\mathcal{C})$ composed with the weak equivalence $\Sigma WTHH^\Gamma(\mathcal{C}) \rightarrow |WTHH^\Gamma(S_\bullet \mathcal{C})|$. \square

Using the alternate models \widetilde{WTC} and \widetilde{WTHH} of Definition 5.9, we get constructions $\widetilde{WTC}^\Gamma(f)$ and $\widetilde{WTHH}^\Gamma(f)$ that admit a cyclotomic trace from K -theory. Because on objects, the map constructed in the proof of Theorem 7.2 agrees with the corresponding map in cofiber sequence on K -theory, we get the following theorem as an immediate consequence.

Theorem 7.3. *For $f: \mathcal{C} \rightarrow \mathcal{D}$ an enriched exact functor, the following diagram commutes.*

$$\begin{array}{ccccccc} K(\mathcal{C}_0) & \longrightarrow & K(\mathcal{D}_0) & \longrightarrow & K(f) & \longrightarrow & K(S_\bullet \mathcal{C}_0) \\ \text{trc} \downarrow & & \text{trc} \downarrow & & \text{trc} \downarrow & & \text{trc} \downarrow \\ \widetilde{WTC}^\Gamma(\mathcal{C}) & \longrightarrow & \widetilde{WTC}^\Gamma(\mathcal{D}) & \longrightarrow & \widetilde{WTC}^\Gamma(f) & \longrightarrow & \widetilde{WTC}^\Gamma(S_\bullet \mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{WTHH}^\Gamma(\mathcal{C}) & \longrightarrow & \widetilde{WTHH}^\Gamma(\mathcal{D}) & \longrightarrow & \widetilde{WTHH}^\Gamma(f) & \longrightarrow & \widetilde{WTHH}^\Gamma(S_\bullet \mathcal{C}) \end{array}$$

Returning to Theorem 7.2, we have the following corollary that allows us to study the cofibers of exact functors in “ THH -theoretic” terms.

Corollary 7.4. *Let $f: \mathcal{A} \rightarrow \mathcal{B}$ and $g: \mathcal{C} \rightarrow \mathcal{D}$ be enriched exact functors. Then the commutative square of cyclotomic spectra on the left is homotopy (co)cartesian.*

$$\begin{array}{ccc} WTHH^\Gamma(\mathcal{B}) & \longrightarrow & WTHH^\Gamma(f) \\ \downarrow & & \downarrow \\ WTHH^\Gamma(\mathcal{C}) & \longrightarrow & WTHH^\Gamma(g \circ f) \end{array} \quad \begin{array}{ccc} WTHH(\mathcal{B}) & \longrightarrow & WTHH(f) \\ \downarrow & & \downarrow \\ WTHH(\mathcal{C}) & \longrightarrow & WTHH(g \circ f) \end{array}$$

If f and g are enhanced exact then the commutative square of cyclotomic spectra on the right is homotopy cartesian.

In the special case when \mathcal{C} is a simplicially enriched Waldhausen subcategory of \mathcal{D} and f is the inclusion, $S_\bullet f$ admits an equivalent but smaller variant where we omit the choices of subquotients.

Definition 7.5. We say that $\mathcal{C} \subset \mathcal{D}$ is a *simplicially enriched Waldhausen subcategory* when $\mathcal{C} \subset \mathcal{D}$ is full as a simplicially enriched category and \mathcal{C}_0 is a Waldhausen subcategory of \mathcal{D}_0 . In this case we define $F_\bullet(\mathcal{D}, \mathcal{C})$ to be the simplicially enriched Waldhausen subcategory of the nerve of the cofibrations in \mathcal{D} whose quotients lie in \mathcal{C} .

Concretely, $F_n(\mathcal{D}, \mathcal{C})$ has as objects the composable sequences of n cofibrations

$$x_0 \rightrightarrows x_1 \rightrightarrows \cdots \rightrightarrows x_n$$

such that x_{i+1}/x_i is an object of \mathcal{C} for all i , with maps the simplicial sets of natural transformations. We have a forgetful functor from $S_\bullet(\mathcal{C} \rightarrow \mathcal{D})$ to $F_\bullet(\mathcal{D}, \mathcal{C})$ that throws away the subquotients, i.e., sending (X, Y) in $S_{n+1}\mathcal{D} \times S_n\mathcal{C}$ to

$$x_{0,1} \rightrightarrows x_{0,2} \rightrightarrows \cdots \rightrightarrows x_{0,n+1}$$

in $F_n(\mathcal{D}, \mathcal{C})$, where $X = (x_{i,j})$. At each simplicial level this map is an equivalence of simplicial Waldhausen categories, and in particular induces a DK-equivalence

$$S_m S_n(\mathcal{C} \rightarrow \mathcal{D}) \longrightarrow S_m F_n(\mathcal{D}, \mathcal{C}).$$

We therefore obtain the following observation, useful in combination with Theorem 7.2.

Proposition 7.6. *For $\mathcal{C} \subset \mathcal{D}$ a simplicially enriched Waldhausen subcategory, the forgetful functor from $S_\bullet(\mathcal{C} \rightarrow \mathcal{D})$ to $F_\bullet(\mathcal{D}, \mathcal{C})$ induces a weak equivalence of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C} \rightarrow \mathcal{D}) \longrightarrow |WTHH^\Gamma(F_\bullet(\mathcal{D}, \mathcal{C}))|.$$

We have the notion of a *closed* simplicially enriched Waldhausen subcategory, which is a simplicially enriched Waldhausen subcategory $\mathcal{A} \subset \mathcal{B}$ where \mathcal{A}_0 is a closed Waldhausen subcategory of \mathcal{B}_0 (i.e., every object of \mathcal{B} weakly equivalent to an object of \mathcal{A} is in \mathcal{A}). When \mathcal{B} is an enhanced simplicially enriched Waldhausen category and $\mathcal{A} \subset \mathcal{B}$ is a closed simplicially enriched Waldhausen subcategory, then \mathcal{A} is also enhanced simplicially enriched. The discussion above then generalizes to show that

$$S_m S_n(\mathcal{C} \rightarrow \mathcal{D}) \longrightarrow S_m F_n(\mathcal{D}, \mathcal{C}).$$

induces an equivalence (and in particular DK-equivalence) on non-connective enrichments. It follows that

$$WTHH(\mathcal{A} \rightarrow \mathcal{B}) \longrightarrow |WTHH(F_\bullet(\mathcal{B}, \mathcal{A}))|$$

is also a weak equivalence of cyclotomic spectra.

8. THE LOCALIZATION THEOREM

The Localization Theorem, called by Waldhausen the “Fibration Theorem”, provides the most important instance of the Cofiber Theorem. Roughly speaking, this theorem states that algebraic K -theory takes quotient sequences of triangulated categories to cofiber sequences of spectra. In this section, we prove versions of this theorem for THH and TC . In the case of the non-connective enrichment, we obtain

a localization sequence equivalent to the one in [4]; in the case of the connective enrichment, we obtain a localization sequence generalizing the one in [13].

For the setup for the Localization Theorem, we take an enhanced simplicially enriched Waldhausen category \mathcal{A} together with an additional subcategory of weak equivalences $v\mathcal{A}_0$ that contains its usual weak equivalences $w\mathcal{A}_0$. We assume that $v\mathcal{A}_0$ satisfies the two-out-of-three properties and the Extension Axiom [28, §1.2], meaning that given a map of cofibration sequences

$$\begin{array}{ccccc} x & \twoheadrightarrow & y & \longrightarrow & y/x \\ v \downarrow & & \downarrow & & \downarrow v \\ x' & \twoheadrightarrow & y' & \longrightarrow & y'/x' \end{array}$$

with the outer maps $x \rightarrow x'$ and $x/y \rightarrow x'/y'$ in $v\mathcal{A}_0$, then the inner map $y \rightarrow y'$ is in $v\mathcal{A}_0$. Finally, recalling that as an enhanced simplicially Waldhausen category, \mathcal{A} admits tensors with contractible simplicial sets, we say that $v\mathcal{A}_0$ is *compatible with cylinders* when for any map $x \rightarrow x'$ in $v\mathcal{A}_0$, the map

$$x \longrightarrow x' \cup_x (x \otimes \Delta[1])$$

is a cofibration in \mathcal{A}_0 , i.e., its quotient is in \mathcal{A} . The category of *v-acyclics* \mathcal{A}_0^v consists of the full subcategory of objects v -equivalent to the trivial object $*$. Under these hypotheses, \mathcal{A}_0^v forms a closed Waldhausen subcategory of \mathcal{A} . Moreover, \mathcal{A}_0^v is *closed under extensions and cofibers in \mathcal{A}_0* , meaning that for a cofibration sequence in \mathcal{A}_0

$$x \twoheadrightarrow y \longrightarrow y/x,$$

if x and either of y or y/x is in \mathcal{A}_0^v , then so is the other. Letting \mathcal{A}^v be the full simplicially enriched subcategory of \mathcal{A} consisting of the objects in \mathcal{A}_0^v , then \mathcal{A}^v forms an enhanced simplicially enriched Waldhausen category with the inclusion functor $\mathcal{A}^v \rightarrow \mathcal{A}$ enhanced exact. We can now state the Localization Theorem.

Theorem 8.1 (Localization Theorem). *With hypotheses and notation as in the previous paragraph, the following commutative squares of cyclotomic spectra are homotopy (co)cartesian.*

$$\begin{array}{ccc} WTHH^\Gamma(\mathcal{A}^v) & \longrightarrow & WTHH^\Gamma(\mathcal{A}^v|v) \\ \downarrow & & \downarrow \\ WTHH^\Gamma(\mathcal{A}) & \longrightarrow & WTHH^\Gamma(\mathcal{A}|v) \end{array} \quad \begin{array}{ccc} WTHH(\mathcal{A}^v) & \longrightarrow & WTHH(\mathcal{A}^v|v) \\ \downarrow & & \downarrow \\ WTHH(\mathcal{A}) & \longrightarrow & WTHH(\mathcal{A}|v) \end{array}$$

Moreover, in each square, the upper right entry is null homotopic through cyclotomic maps. Thus, we have cofiber sequences of cyclotomic spectra,

$$\begin{aligned} WTHH^\Gamma(\mathcal{A}^v) &\longrightarrow WTHH^\Gamma(\mathcal{A}) \longrightarrow WTHH^\Gamma(\mathcal{A}|v) \longrightarrow \Sigma WTHH^\Gamma(\mathcal{A}^v) \\ WTHH(\mathcal{A}^v) &\longrightarrow WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{A}|v) \longrightarrow \Sigma WTHH(\mathcal{A}^v). \end{aligned}$$

Although formally similar in statement and proof, the two localization sequences above are very different in practice. In the case when \mathcal{A} is pretriangulated (which by Corollary 2.12 just means in this context that every object is weakly equivalent to a suspension), the Localization Theorem of [4] identifies the relative term $WTHH(\mathcal{A}|v)$ in the second sequence above as the THH of the triangulated quotient $\mathcal{A}/\mathcal{A}^v$ (for any spectrally enriched model of this quotient).

In the special case when \mathcal{A} is the category of finite cell EKMM R -modules for the S -algebra $R = HA$ for a discrete valuation ring A or $R = ku$ is connective K -theory, we take the v -equivalences $v\mathcal{A}_0$ to be the $R[\beta^{-1}]$ -equivalences, the maps that induce isomorphisms on homotopy groups after inverting β , where β is a uniformizer for A (when $R = HA$) or is the Bott-element (when $R = ku$). Then Theorem 5.14 (proved in Section 9) combined with Theorem 5.13 identify both $WTHH^\Gamma(\mathcal{A})$ and $WTHH(\mathcal{A})$ as $THH(R)$. In the non-connective case, we then have that $WTHH(\mathcal{A}|v)$ is equivalent to $THH(R[\beta^{-1}])$. Calculations show $WTHH(\mathcal{A}^v)$ cannot be equivalent to $THH(R/\beta)$. On the other hand, we will prove a dévissage theorem in Part 4 that identifies $WTHH^\Gamma(\mathcal{A}^v)$ as $THH(R/\beta)$ and calculations show that $WTHH^\Gamma(\mathcal{A}|v)$ cannot be equivalent to $THH(R[\beta^{-1}])$.

Returning to Theorem 8.1, it follows that the analogous squares in the “tilde” models \widetilde{WTHH}^Γ and \widetilde{WTC}^Γ are homotopy (co)cartesian as well, and we get cofiber sequences on \widetilde{WTHH}^Γ and \widetilde{WTC}^Γ . By naturality, the maps in the squares and in the cofiber sequences commute with the cyclotomic trace. For convenient reference, we state this explicitly in the following theorem.

Theorem 8.2. *Under the hypotheses of Theorem 8.1, the following diagram of cofiber sequences commutes.*

$$\begin{array}{ccccccc}
K(\mathcal{A}_0^v) & \longrightarrow & K(\mathcal{A}_0) & \longrightarrow & K(\mathcal{A}_0|v) & \longrightarrow & \Sigma K(\mathcal{A}_0^v) \\
\text{trc} \downarrow & & \text{trc} \downarrow & & \text{trc} \downarrow & & \text{trc} \downarrow \\
\widetilde{WTC}^\Gamma(\mathcal{A}^v) & \longrightarrow & \widetilde{WTC}^\Gamma(\mathcal{A}) & \longrightarrow & \widetilde{WTC}^\Gamma(\mathcal{A}|v) & \longrightarrow & \Sigma \widetilde{WTC}^\Gamma(\mathcal{A}^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{WTHH}^\Gamma(\mathcal{A}^v) & \longrightarrow & \widetilde{WTHH}^\Gamma(\mathcal{A}) & \longrightarrow & \widetilde{WTHH}^\Gamma(\mathcal{A}|v) & \longrightarrow & \Sigma \widetilde{WTHH}^\Gamma(\mathcal{A}^v)
\end{array}$$

We begin the proof of Theorem 8.1 by noting that the category of v -acyclics completely characterizes the v -equivalences $v\mathcal{A}_0$.

Proposition 8.3. *Under the hypotheses of Theorem 8.1, a map $f: x \rightarrow y$ is in $v\mathcal{A}_0$ if and only if the homotopy cofiber*

$$Cf = y \cup_x (x \otimes \Delta[1]) \cup_x *$$

is in \mathcal{A}_0^v .

Proof. Let $Mf = y \cup_x (x \otimes \Delta[1])$ so that $Cf = Mf/x$. The map $Mf \rightarrow y$ is a weak equivalence (and so in particular a v -equivalence) and the composite map $x \rightarrow Mf \rightarrow y$ is f and so the inclusion of x in Mf is in $v\mathcal{A}_0$ if and only if f is. Consider the commutative diagram of cofiber sequences

$$\begin{array}{ccccc}
x & \xrightarrow{=} & x & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
x & \xrightarrow{=} & Mf & \longrightarrow & Cf.
\end{array}$$

By the Gluing Axiom, Cf is in \mathcal{A}_0^v when $x \rightarrow Mf$ is in $v\mathcal{A}_0$. By the Extension Axiom $x \rightarrow Mf$ is in $v\mathcal{A}_0$ when Cf is in \mathcal{A}_0^v . \square

Let $\bar{v}\mathcal{A}_0 = v\mathcal{A}_0 \cap \text{co}\mathcal{A}_0$ denote the subcategory of \mathcal{A}_0 consisting of the maps that are both cofibrations and v -equivalences. The previous proposition implies that $\bar{v}\mathcal{A}_0$ consists of those cofibrations whose quotients are v -acyclic. It follows

that $F_\bullet(\mathcal{A}, \mathcal{A}^v) = \bar{v}_\bullet \mathcal{A}$, and applying Corollary 7.4 and Proposition 7.6, we get homotopy (co)cartesian squares

$$\begin{array}{ccc} WTHH^\Gamma(\mathcal{A}^v) \longrightarrow |WTHH^\Gamma(\bar{v}_\bullet \mathcal{A}^v)| & WTHH(\mathcal{A}^v) \longrightarrow |WTHH(\bar{v}_\bullet \mathcal{A}^v)| \\ \downarrow & \downarrow & \downarrow \\ WTHH^\Gamma(\mathcal{A}) \longrightarrow |WTHH^\Gamma(\bar{v}_\bullet \mathcal{A})| & WTHH(\mathcal{A}) \longrightarrow |WTHH(\bar{v}_\bullet \mathcal{A})|. \end{array}$$

We now have what we need to prove Theorem 8.1.

Proof of Theorem 8.1. To obtain the homotopy (co)cartesian squares, we just need to see that the maps

$$WTHH^\Gamma(\bar{v}_\bullet \mathcal{A}) \longrightarrow WTHH^\Gamma(v_\bullet^M \mathcal{A}) \quad \text{and} \quad WTHH(\bar{v}_\bullet \mathcal{A}) \longrightarrow WTHH(v_\bullet^M \mathcal{A})$$

are weak equivalences. The inclusion of $|\bar{v}_p S_q \mathcal{A}|$ in $|v_p^M S_q \mathcal{A}|$ is a DK-embedding and an easy mapping cylinder argument shows that it is a DK-equivalence.

It follows that $WTHH^\Gamma(\mathcal{A}^v|v)$ and $WTHH^\Gamma(\mathcal{A}^v|v)$ are weakly equivalent as cyclotomic spectra to the trivial spectrum, and to produce a null homotopy through cyclotomic maps is not much more work. The simplicial object $v_\bullet^M \mathcal{A}^v$ has an extra degeneracy which on objects inserts the trivial map at the start of the chain of maps. On maps, we use the unique (constant trivial) homotopy on any subsimplex that has the new trivial object as one of its vertices. \square

9. THE SPHERE THEOREM

In this section, we state versions of Waldhausen’s “Sphere Theorem” for the THH of Waldhausen categories, which we prove in the next section. These theorems allow us to deduce the important consistency result that all the different models for the THH of the finite-cell modules over an EKMM S -algebra or a simplicial ring agree (Theorem 5.14 above). Before stating a precise theorem, we need two definitions,

Definition 9.1. Let \mathcal{C} be a simplicially tensored Waldhausen category. We say that \mathcal{C} is *stable* when:

- (i) Every object of \mathcal{C} is weakly equivalent to a suspension, and
- (ii) For all objects x and y in \mathcal{C} , the suspension map $\mathcal{C}(x, y) \rightarrow \mathcal{C}(\Sigma x, \Sigma y)$ is a weak equivalence.

We say that \mathcal{C} is *almost stable* when it satisfies just condition (ii).

As observed in Corollary 2.12, the first condition implies that the non-connective spectral category \mathcal{C}^S is pretriangulated, and its homotopy category $\pi_0 \mathcal{C}^S$ is triangulated. The second condition implies that the homotopy category $\pi_0 \mathcal{C}^S$ coincides with the homotopy category $\pi_0 \mathcal{C}$ and also that the connective spectral enrichment $\mathcal{C}^\Gamma(x, y)$ is the connective cover of the non-connective spectral enrichment $\mathcal{C}^S(x, y)$ (Proposition 2.10). Combined with the fact that the mapping simplicial sets $\mathcal{C}(x, y)$ are Kan complexes (and that weak equivalences in \mathcal{C} are homotopy equivalences in the obvious sense), this puts all the basic tools and techniques of homotopy theory and stable homotopy theory at our disposal.

In the stable case the hypotheses we need for the Sphere Theorem greatly simplify and so we will explore that case first. In addition to the stability assumptions above, we need to assume that \mathcal{C} is generated by connective objects in the following sense.

Definition 9.2. Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category. A *connective class* Q in \mathcal{C} is a set of objects of \mathcal{C} such that for any a, b in Q , $\mathcal{C}^S(a, b)$ is connective. If \mathcal{C} is stable, then we say that Q is *generating* if the smallest triangulated subcategory of the triangulated category $\pi_0\mathcal{C}^S$ that contains Q is all of $\pi_0\mathcal{C}^S$.

See Definition 9.9 for the definition of generating when \mathcal{C} is almost stable. In this terminology, we prove the following theorem, the THH analogue of Waldhausen’s Sphere Theorem for the stable case.

Theorem 9.3 (Sphere Theorem, Stable Version). *Let \mathcal{C} be a stable simplicially tensored Waldhausen category and assume that \mathcal{C} has a generating connective class Q . Then the canonical cyclotomic maps are weak equivalences*

$$WTHH^\Gamma(\mathcal{C}) \xrightarrow{\sim} WTHH(\mathcal{C}) \xleftarrow{\sim} THH(\mathcal{C}^S) \xleftarrow{\sim} THH(Q^S).$$

Here Q^S denotes the full spectral subcategory of \mathcal{C}^S on the objects of Q .

We state the following corollary for ease of reference and citation; it is one case of Theorem 5.14.

Corollary 9.4. *Let R be a connective EKMM S -algebra and \mathcal{C}_R the category of finite cell R -modules. Then the canonical cyclotomic maps*

$$WTHH^\Gamma(\mathcal{C}_R) \xrightarrow{\sim} WTHH(\mathcal{C}_R) \xleftarrow{\sim} THH(\mathcal{C}_R^S) \xleftarrow{\sim} THH(R)$$

are weak equivalences. Here $THH(R)$ denotes the usual Bökstedt model of THH of a particular symmetric ring spectrum (or FSP) equivalent to R .

The corollary follows by taking the connective class Q to be the singleton set containing the one object S_R , the “cell zero sphere R -module” [11, III.2]. Then $THH(Q^S)$ coincides with THH of the symmetric ring spectrum

$$F = Q^S(S_R, S_R) = \mathcal{C}_R^S(S_R, S_R).$$

Concretely, this has n -th space

$$F(n) = \mathcal{C}_R(S_R, S_R \wedge S^n),$$

and multiplication induced by composition. We can identify this as the symmetric ring spectrum (or “FSP defined on spheres”) obtained from the FSP $\mathbf{F}(-) = \mathcal{C}_R(S_R, S_R \wedge -)$ by restricting to spheres $F(n) = \mathbf{F}(S^n)$.

Another symmetric ring spectrum derives from the general theory of [21]; writing \mathcal{M}_S for the category of S -modules, this has spaces $\Phi(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, R)$ and multiplication induced by smash product together with the multiplication on R . Experts know how to compare these symmetric ring spectra and therefore their THH , TR , and TC spectra. Briefly, noting that $S_R = R \wedge S_S$, we construct a third symmetric ring spectrum Φ' that lies between them. Φ' has spaces

$$\begin{aligned} \Phi'(n) &= \mathcal{C}_R(S_R \wedge_S (S_S^{-1} \wedge S^1)^{(n)}, S_R \wedge S^n) \\ &\cong \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_R(S_R, S_R \wedge S^n)) \end{aligned}$$

and multiplication induced both by smash product (on the $(S_S^{-1} \wedge S^1)^{(n)}$ factors) and composition (on the $F_R(S_R, S_R \wedge S^n)$ factors). We have a weak equivalence of symmetric ring spectra from F to Φ' given by

$$F(n) = \mathcal{C}_R(S_R, S_R \wedge S^n) \longrightarrow \mathcal{C}_R(S_R \wedge_S (S_S^{-1} \wedge S^1)^{(n)}, S_R \wedge S^n) = \Phi'(n)$$

induced by the collapse map $S_S^{-1} \wedge S^1 \rightarrow S$; the induced map $F(n) \rightarrow \Phi'(n)$ is a weak equivalence of simplicial sets for all n . We have a weak equivalence of symmetric ring spectra from Φ to Φ' given by

$$\begin{aligned} \Phi(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, R) &\longrightarrow \mathcal{M}_S((S_S^{-1})^{(n)}, F_R(S_R, S_R)) \\ &\longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_R(S_R, S_R \wedge S^n)) \cong \Phi'(n) \end{aligned}$$

induced by the unit map $R \rightarrow F_R(S_R, S_R)$ (which arises from the extra R action on $S_R = R \wedge S_S$); again, this is a weak equivalence of simplicial sets for all n . For convenience, we state these remarks as a proposition.

Proposition 9.5. *The symmetric ring spectrum in Corollary 9.4 is weakly equivalent to the symmetric ring spectrum obtained from the EKMM S -algebra R by [21].*

For the other half of Theorem 5.14, we need to treat the almost stable case. This requires introducing the following subcategories of \mathcal{C} associated to a connective class \mathcal{Q} .

Notation 9.6. Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category and let \mathcal{Q} be a connective class. Write \mathcal{Q} for the smallest closed Waldhausen category of \mathcal{C} containing \mathcal{Q} . For $n \geq 0$, let $\Sigma^n \mathcal{Q}$ denote the full subcategory of \mathcal{C} containing all objects weakly equivalent to $\Sigma^n x$ for x in \mathcal{Q} . Let $\Sigma^{-n} \mathcal{Q}$ denote the full subcategory of \mathcal{C} containing all x such that $\Sigma^n x$ is in \mathcal{Q} .

We note that the subcategories $\Sigma^n \mathcal{Q}$ are themselves connective classes and closed Waldhausen subcategories.

Proposition 9.7. *Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category and let \mathcal{Q} be a connective class. Then $\Sigma^n \mathcal{Q}$ is a connective class and closed Waldhausen subcategory for all $n \in \mathbb{Z}$.*

Proof. We begin by showing that \mathcal{Q} is a connective class; stability hypothesis (ii) then shows that $\Sigma^n \mathcal{Q}$ is a connective class for all n . Let \mathcal{Q}_0 be the collection of objects of \mathcal{C} weakly equivalent to finite coproducts of objects in \mathcal{Q} , and inductively let \mathcal{Q}_n be the collection of objects of \mathcal{C} that are weakly equivalent to finite coproducts of homotopy pushouts $y \cup_x (x \otimes \Delta[1]) \cup_x z$ where $x, y, z \in \mathcal{Q}_{n-1}$ and $y \cup_x (x \otimes \Delta[1]) \cup_x * \in \mathcal{Q}_{n-1}$. If we regard $\bigcup \mathcal{Q}_n$ as the full subcategory of \mathcal{C} of objects in \mathcal{Q}_n for some n , it is then clear that $\mathcal{Q} = \bigcup \mathcal{Q}_n$ is the smallest closed Waldhausen subcategory of \mathcal{C} containing \mathcal{Q} . To show that \mathcal{Q} is a connective class, it suffices to show that for x, y in \mathcal{Q}_n , $\mathcal{C}^S(x, y)$ is connective, which we do by induction. We know that x is weakly equivalent to a finite coproduct of homotopy pushouts of objects in \mathcal{Q}_{n-1} along maps whose homotopy cofiber is also in \mathcal{Q}_{n-1} . Looking at the long exact sequence of homotopy groups from the fibration sequence in Proposition 2.11, we then see that $\mathcal{C}(x, z)$ is connective for all z in \mathcal{Q}_{n-1} . Using the same fact about y and the long exact sequence of homotopy groups from the cofibration sequence in Proposition 2.11, we see that $\mathcal{C}(x, y)$ is connective.

By definition $\Sigma^0 \mathcal{Q} = \mathcal{Q}$ is a closed Waldhausen subcategory and it follows that $\Sigma^n \mathcal{Q}$ is a closed Waldhausen subcategory for $n < 0$ since suspension preserves homotopy pushouts. Let $n > 0$ and suppose $f: x \rightarrow y$ is a cofibration in \mathcal{C} such that x, y , and $y/x \simeq Cf$ are all in $\Sigma^n \mathcal{Q}$. Then we can find x' and y' in \mathcal{Q} and weak equivalences $\Sigma^n x' \rightarrow x$ and $\Sigma^n y' \rightarrow y$. By stability hypothesis (ii) and the fact

that the mapping spaces in \mathcal{C} are Kan complexes, we can find a map $f': x' \rightarrow y'$ such that the diagram

$$\begin{array}{ccc} \Sigma^n x' & \xrightarrow{\Sigma^n f'} & \Sigma^n y' \\ \sim \downarrow & & \sim \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

commutes up to homotopy. Choosing a homotopy, we get a weak equivalence $C\Sigma^n f' \rightarrow Cf$. Then y/x is weakly equivalent to $\Sigma^n Cf$ and it follows (again applying stability hypothesis (ii)) that Cf is in \mathcal{Q} . For any map $x \rightarrow z$ with z in $\Sigma^n \mathcal{Q}$, we can choose a compatible map $x' \rightarrow z'$ (for some z' with $\Sigma^n z' \simeq z$) such that the pushout $w = z \cup_x y$ is weakly equivalent to Σ^n of the homotopy pushout $w' = z' \cup_{x'} (x' \otimes \Delta[1]) \cup_{x'} y'$. Since \mathcal{Q} is a closed Waldhausen subcategory of \mathcal{C} , w' is in \mathcal{Q} , and it follows that w is in $\Sigma^n \mathcal{Q}$. This shows that $\Sigma^n \mathcal{Q}$ is a closed Waldhausen subcategory of \mathcal{C} . \square

We use the subcategories $\Sigma^n \mathcal{Q}$ to define what it means for a connective class to be generating in the almost stable case. For this, we need the following technical definitions.

Definition 9.8. Given a class A of objects of a Waldhausen category \mathcal{C} , we say that an object x of \mathcal{C} is *finitely cellularly built from* A if we can find a sequence of objects x_0, x_1, \dots, x_n of \mathcal{C} that fit into pushout squares

$$\begin{array}{ccc} a_j & \xrightarrow{\quad} & b_j \\ \downarrow & & \downarrow \\ x_j & \xrightarrow{\quad} & x_{j+1} \end{array}$$

where x_n is weakly equivalent to x , $x_0 = *$, and for each j , a_j is in A , b_j is contractible (weakly equivalent to $*$), and $a_j \rightarrow b_j$ is a cofibration.

The concept of “finitely cellularly built from” above differs from other notions of “built from” in other contexts. Note in particular that an object of A is not necessarily finitely cellularly built from A . However, suspensions of objects of A are finitely cellularly built from A , for example.

Definition 9.9. Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category and Q a connective class. We say that Q is *generating* if every object of \mathcal{C} is finitely cellularly built from $\bigcup \Sigma^n \mathcal{Q}$.

The following proposition clarifies the relationship between the notions of generating given in Definitions 9.2 and 9.9.

Proposition 9.10. *Let \mathcal{C} be a stable simplicially tensored Waldhausen category and Q a connective class. Then Q is generating in the sense of Definition 9.2 if and only if it is generating in the sense of Definition 9.9.*

Proof. Since the triangulated subcategory generated by Q contains $\bigcup \Sigma^n \mathcal{Q}$ (cf. the proof of Proposition 9.7), one direction is clear. We must show that if Q is generating in the sense of Definition 9.2, then it is generating in the sense of Definition 9.9. Let $C_0 = \bigcup \Sigma^n \mathcal{Q}$, and inductively let C_n be the collection of objects of \mathcal{C} that are weakly equivalent to the homotopy cofiber of a map between objects of C_{n-1} . We note that the C_n are closed under suspension and desuspension and

that the objects of C_0 are finitely cellularly built from C_0 (since each is equivalent to the suspension of an object of C_0). Generating in the sense of Definition 9.2 implies that $\mathcal{C} = \bigcup C_n$, so it suffices to show by induction that all objects of C_n are finitely cellularly built from C_0 . Given $f: x \rightarrow y$ with x, y in C_{n-1} , we need to show that $z = Cf = y \cup_x (x \otimes \Delta[1]) \cup_x *$ is finitely cellularly built from C_0 . Replacing z with a weakly equivalent object, we can assume without loss of generality that x and y are isomorphic rather than just weakly equivalent to an iterated pushout. Then we build z by first building y and then gluing $Cb_j = (b_j \otimes \Delta[1]) \cup_{b_j} *$ along $a'_j = b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} *$ where $a_j \rightarrow b_j$ build x . Since Cb_j is contractible and a'_j is weakly equivalent to Σa_j , this shows that z is finitely cellularly built from C_0 . \square

The following theorem now generalizes Theorem 9.3 to the almost stable case.

Theorem 9.11 (Sphere Theorem). *Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category and assume that \mathcal{C} has a generating connective class Q . Then the canonical cyclotomic maps are weak equivalences*

$$WTHH^\Gamma(\mathcal{C}) \xrightarrow{\sim} WTHH(\mathcal{C}) \xleftarrow{\sim} THH(\mathcal{C}^S) \xleftarrow{\sim} THH(Q^S).$$

where Q^S denotes the full spectral subcategory of \mathcal{C}^S on the objects of Q .

We now have the other half of Theorem 5.14 as a corollary.

Corollary 9.12. *Let A be a simplicial ring, let \mathcal{C}_A be the category of finite cell A -modules and let \mathcal{P}_A be the category of finite cell A -modules built out of finitely generated projective A -modules. Then the canonical cyclotomic maps*

$$\begin{array}{ccccccc} WTHH^\Gamma(\mathcal{C}_A) & \xrightarrow{\sim} & WTHH(\mathcal{C}_A) & \xleftarrow{\sim} & THH(\mathcal{C}_A^S) & \xleftarrow{\sim} & THH(A) \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow & & \\ WTHH^\Gamma(\mathcal{P}_A) & \xrightarrow{\sim} & WTHH(\mathcal{P}_A) & \xleftarrow{\sim} & THH(\mathcal{P}_A^S) & & \end{array}$$

are weak equivalences.

The vertical arrows are weak equivalences by [4, 4.11] since every object of \mathcal{P}_A or $S_n \mathcal{P}_A$ is a direct summand of an object of \mathcal{C}_A or $S_n \mathcal{C}_A$. We get the top row from Theorem 9.11 taking the connective class Q to be the singleton set containing the object A , which is clearly generating. The symmetric spectrum $Q^S(A, A) = \mathcal{C}^S(A, A)$ is just the usual symmetric ring spectrum constructed from A .

We have one last version of the Sphere Theorem, which is closer in spirit to Waldhausen's Sphere Theorem for K -theory. It also has the technical advantage of being stated purely in terms of the connective enrichments.

Theorem 9.13 (Sphere Theorem, Alternate Version). *Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category, let Q be a generating connective class, and let \mathcal{Q} be the smallest closed Waldhausen subcategory of \mathcal{C} containing Q . The inclusion of \mathcal{Q} into \mathcal{C} induces a weak equivalence $WTHH^\Gamma(\mathcal{Q}) \rightarrow WTHH^\Gamma(\mathcal{C})$.*

The previous theorem is equivalent to Theorem 9.11, but to see this, we need more information about the categories $S_\bullet \mathcal{Q}$ implicit in the statement. The following proposition has everything we need for the comparison, plus what we need for the proofs in the next section.

Proposition 9.14. *Let \mathcal{C} be an almost stable simplicially tensored Waldhausen category and let \mathcal{Q} be a connective class. Then $S_n\mathcal{C}$ is an almost stable simplicially tensored Waldhausen category, and $S_n\mathcal{Q}$ is a closed Waldhausen subcategory and a connective class; moreover, $S_n\Sigma^m\mathcal{Q} = \Sigma^m S_n\mathcal{Q}$. If \mathcal{Q} is generating, then so is $S_n\mathcal{Q}$.*

Proof. We saw in Proposition 3.3 that $S_n\mathcal{C}$ is simplicially tensored; the fact that the tensor on $S_n\mathcal{C}$ is objectwise on the diagram and the formula (3.2) for the mapping spaces of $S_n\mathcal{C}$ prove that $S_n\mathcal{C}$ is almost stable. Since \mathcal{Q} is a closed Waldhausen subcategory of \mathcal{C} , $S_n\mathcal{Q}$ is a closed Waldhausen subcategory of $S_n\mathcal{C}$. Again, the formula (3.2) shows that the mapping spectra $S_n\mathcal{Q}^S(A, B)$ are connective. It is clear that $S_n\Sigma^m\mathcal{Q} = \Sigma^m S_n\mathcal{Q}$ since both categories are the functor categories whose objects are the sequences starting with $*$ of n composable cofibrations in \mathcal{C} between objects in $\Sigma^m\mathcal{Q}$ together with choices of quotients which also must be in $\Sigma^m\mathcal{Q}$.

Now assume that \mathcal{Q} is generating; it remains to show that $S_n\mathcal{Q}$ is generating. For $n = 2$, a typical object of $S_2\mathcal{C}$ is of the form $Z = [x \twoheadrightarrow y \twoheadrightarrow z]$ for objects x, y, z in \mathcal{C} . Replacing Z with a weakly equivalent object, we can assume without loss of generality that x and y are isomorphic rather than just weakly equivalent to an iterated pushout in Definition 9.8. Clearly the objects

$$[x \twoheadrightarrow x \twoheadrightarrow *] \quad \text{and} \quad [* \twoheadrightarrow y \twoheadrightarrow y]$$

can be finitely cellularly built using pushouts of objects of the same form. We can then build

$$Z' = [x \twoheadrightarrow y \cup_x (x \otimes \Delta[1]) \twoheadrightarrow y \cup_x (x \otimes \Delta[1]) \cup_x *]$$

by first building $[x \twoheadrightarrow x \amalg y \twoheadrightarrow y]$ and then using pushouts over maps of the form

$$\begin{aligned} [* \twoheadrightarrow b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} b_j &= b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} b_j] \\ &\longrightarrow [* \twoheadrightarrow b_j \otimes \Delta[1] = b_j \otimes \Delta[1]] \end{aligned}$$

where $a_j \rightarrow b_j$ are the cells building x . Similar observations apply for $n > 2$. \square

Now Theorems 9.11 and 9.13 are easily seen to be equivalent by looking at the following diagram.

$$\begin{array}{ccccccc} \Omega|THH(S_\bullet \mathcal{Q}^\Gamma)| & \longrightarrow & \Omega|THH(S_\bullet \mathcal{C}^\Gamma)| & & & & \\ \sim \downarrow & & \downarrow & & & & \\ \Omega|THH(S_\bullet \mathcal{Q}^S)| & \xrightarrow{\sim} & \Omega|THH(S_\bullet \mathcal{C}^S)| & \xleftarrow{\sim} & THH(\mathcal{C}^S) & \xleftarrow{\sim} & THH(\mathcal{Q}^S). \end{array}$$

The lefthand vertical map is a weak equivalence since each map $S_n\mathcal{Q}^\Gamma \rightarrow S_n\mathcal{Q}^S$ is a weak equivalence (and in particular DK-equivalence) of spectral categories by Propositions 9.7 and 9.14, while the bottom horizontal maps are weak equivalences by [4, 4.12] (for the first and third maps) and Theorem 5.13 (for the middle map). Theorem 9.11 then amounts to the assertion that the righthand vertical map is a weak equivalence while Theorem 9.13 is the assertion that the top horizontal map is a weak equivalence.

10. PROOF OF THE SPHERE THEOREM

This section contains the proof of Theorem 9.13. We fix the almost stable simplicially tensored Waldhausen category \mathcal{C} and the generating connective class Q , letting \mathcal{Q} and $\Sigma^n \mathcal{Q}$ be as in Notation 9.6. Just as in Waldhausen's argument [28, §1.7] we need to introduce a Waldhausen category of CW complexes built out of cells based on Q .

Definition 10.1. A Q -CW complex is a filtered object X in \mathcal{C}

$$\cdots \longrightarrow x_n \longrightarrow x_{n+1} \longrightarrow \cdots$$

indexed on the integers, where the arrows $x_n \rightarrow x_{n+1}$ are cofibrations for all n and such that for some $N \geq 0$, the following conditions hold:

- (i) $x_n = *$ for $n \leq -N$,
- (ii) $x_n = x_{n+1}$ for $n \geq N$, and
- (iii) For all n , the quotient x_{n+1}/x_n is an object in $\Sigma^{n+1} \mathcal{Q}$.

We call x_N the *underlying object* of X in \mathcal{C} . Let $CW_Q \mathcal{C}$ denote the category whose objects are the Q -CW complexes and whose maps are the maps of the underlying objects in \mathcal{C} . We say that a Q -CW complex X is connective if $x_n = *$ for $n < 0$, and denote the full subcategory of connective Q -CW complexes by $CW_Q \mathcal{C}_{[0, \infty)}$. More generally, for I an interval in \mathbb{Z} , write $CW_Q \mathcal{C}_I$ for the full subcategory of Q -CW complexes X with $x_n = *$ whenever n is less than the elements of I and $x_n = x_{n+1}$ whenever $n+1$ is greater than the elements of I .

We define the mapping spectra in $CW_Q \mathcal{C}^\Gamma$ and $CW_Q \mathcal{C}^S$ as the mapping spectra of the underlying objects in \mathcal{C}^Γ and \mathcal{C}^S , respectively. For the Waldhausen category structure, we use the following definition.

Definition 10.2. A *cellular map* of Q -CW complexes $X \rightarrow Y$ consists of compatible maps $x_n \rightarrow y_n$ for all n . A cellular map is a *cellular cofibration* when each map $x_n \cup_{x_{n-1}} y_{n-1} \rightarrow y_n$ is a cofibration in \mathcal{C} and the induced map $x_n/x_{n-1} \rightarrow y_n/y_{n-1}$ is a cofibration in $\Sigma^n \mathcal{Q}$.

An easy check of the definitions then proves the following proposition.

Proposition 10.3. *The category of Q -CW complexes and cellular maps forms a Waldhausen category with cofibrations the cellular cofibrations of Definition 10.2 and weak equivalences the weak equivalences of the underlying objects in \mathcal{C} . For I an interval in \mathbb{Z} , the subcategory $CW_Q \mathcal{C}_I$ forms a Waldhausen subcategory (though not closed a closed one).*

Since $S_n \mathcal{Q}$ is a connective class, we also have the category of $S_n \mathcal{Q}$ -CW complexes in $S_n \mathcal{C}$. When we restrict to the subcategories of cellular maps, both $S_n(CW_Q \mathcal{C})$ and $CW_{S_n \mathcal{Q}} S_n \mathcal{C}$ are subcategories of the category of functors $\text{Ar}[n] \times \mathbb{Z} \rightarrow \mathcal{C}$ (where the category \mathbb{Z} is the ordered set of integers). An easy check of the definitions then shows that these categories coincide. More generally, for I an interval in \mathbb{Z} , the cellular maps in $S_n(CW_Q \mathcal{C}_I)$ and $(CW_{S_n \mathcal{Q}} S_n \mathcal{C})_I$ are the same subcategory of functors $\text{Ar}[n] \times I \rightarrow \mathcal{C}$. Expanding to all maps in $S_n(CW_Q \mathcal{C})$ and $CW_{S_n \mathcal{Q}} S_n \mathcal{C}$, and looking at the cofibrations and weak equivalences, we get the following proposition.

Proposition 10.4. *The Waldhausen categories $S_n(CW_Q \mathcal{C})$ and $CW_{S_n \mathcal{Q}} S_n \mathcal{C}$ are canonically isomorphic. For any interval I in \mathbb{Z} , the Waldhausen categories $S_n(CW_Q \mathcal{C}_I)$ and $(CW_{S_n \mathcal{Q}} S_n \mathcal{C})_I$ are canonically isomorphic.*

Because we need to restrict to cellular maps to obtain a Waldhausen category, the category $CW_Q\mathcal{C}$ does not fit into our usual framework of simplicially enriched Waldhausen categories (as the familiar example of CW complexes in spaces demonstrates). Instead, thinking of Q -CW complexes as objects of \mathcal{C} with extra structure, we assign mapping spectra by looking at the underlying objects. We use the following notation.

Notation 10.5. Let $S_n(CW_Q\mathcal{C})^\Gamma$ denote the spectral category whose objects are the objects of $S_n(CW_Q\mathcal{C})$ and whose mapping spectra are the mapping spectra of the underlying objects in $S_n\mathcal{C}^\Gamma$. For I an interval in \mathbb{Z} , we define $S_n(CW_Q\mathcal{C}_I)^\Gamma$ analogously.

As an alternate take on this notation, we note that under the canonical isomorphism of Proposition 10.4, we get the identification of spectral categories

$$S_n(CW_Q\mathcal{C}_I)^\Gamma = (CW_{S_n\mathcal{Q}}S_n\mathcal{C}_I)^\Gamma$$

As a first reduction of Theorem 9.13, we have the following observation. In it, the “forgetful functor” is the functor that takes a $S_n\mathcal{Q}$ -CW complex to its underlying object of $S_n\mathcal{C}$.

Proposition 10.6. *For any n , the forgetful functor $S_n(CW_Q\mathcal{C})^\Gamma \rightarrow S_n\mathcal{C}^\Gamma$ is a DK-equivalence.*

Proof. Using the identification of $S_n(CW_Q\mathcal{C})$ as $CW_{S_n\mathcal{Q}}S_n\mathcal{C}$, it suffices to show that for the arbitrary almost stable simplicially tensored Waldhausen category \mathcal{C} and generating connective class Q the forgetful functor $CW_Q\mathcal{C}^\Gamma \rightarrow \mathcal{C}^\Gamma$ is a DK-equivalence. By definition of the mapping spectra, it is a DK-embedding, and so we just need to show that every object of \mathcal{C} is weakly equivalent to the underlying object of a Q -CW complex. Since Q is generating, and $*$ is the underlying object of a Q -CW complex, it suffices to show that if y is the underlying object of a Q -CW complex Y , then $x = y \cup_a b$ is weakly equivalent to the underlying object of a Q -CW whenever a is in $\Sigma^n\mathcal{Q}$, b is contractible, and $a \rightarrow b$ is a cofibration. Using the cofibration sequence of Proposition 2.11 and stability hypothesis (ii), we see that we have homotopy fibration sequences

$$\mathcal{C}(a, y_m) \longrightarrow \mathcal{C}(a, y_{m+1}) \longrightarrow \mathcal{C}(a, y_{m+1}/y_m)$$

for all m . Since y_{m+1}/y_m is in $\Sigma^{m+1}\mathcal{Q}$, for $m \geq n$ we have that $\pi_0\mathcal{C}(a, y_{m+1}/y_m) = 0$ and every map from a to y_{m+1} lifts up to homotopy to a map $a \rightarrow y_m$. Thus, the map $a \rightarrow y$ lifts up to homotopy to a map $a \rightarrow y_n$. Let X be the Q -CW complex

$$X = (\cdots \longrightarrow y_{n-1} \longrightarrow y_n \cup_a b \longrightarrow y_{n+1} \cup_a b \longrightarrow \cdots),$$

Then the underlying object of X is weakly equivalent to x . \square

It follows from Proposition 10.6 that the map

$$THH(S_n(CW_Q\mathcal{C})^\Gamma) \longrightarrow THH(S_n\mathcal{C}^\Gamma)$$

is a weak equivalence. The next step is to compare the subcategory of connective objects. The cone and suspension functor on \mathcal{C} extend to cone and suspension functors of Q -CW complexes in the usual way: for a Q -CW complex X , let CX be the Q -CW complex with n -th object $x_n \cup_{x_{n-1}} Cx_{n-1}$. The inclusion of X in CX is a cellular cofibration and ΣX is its quotient. The Additivity Theorem and Corollary 6.2 generalize to the context of $THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)$ to show that the

self-map of $|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$ induced by C coincides (in the stable category) with the sum of the identity and the map induced by Σ . Since C induces the trivial map, it follows that Σ induces the map $-\text{id}$, and in particular is a weak equivalence. The analogous observations apply to $\mathcal{C}_{[0,\infty)}$, showing that suspension induces a weak equivalence on $|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$ and on $|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$. Taking the homotopy colimit of the maps induced by suspension, we see that the inclusions

$$\begin{aligned} |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)| &\longrightarrow \text{hocolim}_\Sigma |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)| \\ |THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| &\longrightarrow \text{hocolim}_\Sigma |THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \end{aligned}$$

are weak equivalences. We use this observation in the proof of the following proposition.

Proposition 10.7. *The inclusion*

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$$

is a weak equivalence.

Proof. By the preceding observations, it suffices to prove that the map

$$\text{hocolim}_\Sigma THH(S_n(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma) \longrightarrow \text{hocolim}_\Sigma THH(S_n(CW_Q\mathcal{C})^\Gamma)$$

is a weak equivalence for each n . Again using the fact that \mathcal{C} and Q are arbitrary, it suffices to consider the case $n = 1$. Let $CW_Q^\Sigma\mathcal{C}^\Gamma$ be the spectrally enriched category where an object is an ordered pair (X, m) where X is a Q -CW complex and m is a non-negative integer; for mapping spectra, we let

$$CW_Q^\Sigma\mathcal{C}^\Gamma((X, m), (Y, n)) = \text{colim}_{k \geq \max(m, n)} \mathcal{C}^\Gamma(\Sigma^{k-m}X, \Sigma^{k-n}Y).$$

(Composition is induced levelwise in the colimit system after taking k large enough.) Let $CW_Q^\Sigma\mathcal{C}_{[0,\infty)}^\Gamma$ be the full subcategory of $CW_Q^\Sigma\mathcal{C}^\Gamma$ consisting of the objects (X, m) with X connective. Opening up the construction of THH , it is straightforward to construct canonical maps

$$\begin{aligned} \text{hocolim}_\Sigma THH(CW_Q\mathcal{C}^\Gamma) &\longrightarrow THH(CW_Q^\Sigma\mathcal{C}^\Gamma) \\ \text{hocolim}_\Sigma THH(CW_Q\mathcal{C}_{[0,\infty)}^\Gamma) &\longrightarrow THH(CW_Q^\Sigma\mathcal{C}_{[0,\infty)}^\Gamma) \end{aligned}$$

and check that they are weak equivalences. The inclusion of $CW_Q^\Sigma\mathcal{C}_{[0,\infty)}^\Gamma$ in $CW_Q^\Sigma\mathcal{C}^\Gamma$ is a DK-equivalence, and so also induces a weak equivalence on THH . \square

The previous two propositions show that the map

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \longrightarrow |THH(S_\bullet\mathcal{C}^\Gamma)|$$

is a weak equivalence, reducing the proof of Theorem 9.13 to showing that the map

$$|THH(S_\bullet\mathcal{Q}^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$$

is a weak equivalence. This is an easy consequence of the following lemma.

Lemma 10.8. *For every $n \geq 1$, the inclusion of $CW_Q\mathcal{C}_{[0,n-1]}$ in $CW_Q\mathcal{C}_{[0,n]}$ induces a weak equivalence*

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n-1]})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C}_{[0,n]})^\Gamma)|.$$

Proof of Theorem 9.13 from Lemma 10.8. The lemma implies that the maps in the homotopy colimit system

$$\mathrm{hocolim}_n |THH(S_\bullet(CW_Q \mathcal{C}_{[0,n]})^\Gamma)|$$

are all weak equivalences. By inspection we see that the canonical map from the homotopy colimit to $|THH(S_\bullet(CW_Q \mathcal{C}_{[0,\infty)})^\Gamma|$ is a weak equivalence. It follows that the map

$$|THH(S_\bullet \mathcal{Q}^\Gamma)| = |THH(S_\bullet(CW_Q \mathcal{C}_{[0,0]})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q \mathcal{C})^\Gamma)|$$

is a weak equivalence. Composing with the weak equivalence

$$|THH(S_\bullet(CW_Q \mathcal{C})^\Gamma)| \longrightarrow |THH(S_\bullet \mathcal{C}^\Gamma)|$$

above and applying Ω , we see that the map $WTHH^\Gamma(\mathcal{Q}) \rightarrow WTHH^\Gamma(\mathcal{C})$ is a weak equivalence. \square

The remainder of the section is devoted to the proof of Lemma 10.8. The argument is somewhat roundabout, requiring the introduction of the spectral categories $S_k(CW_Q \mathcal{C}_{[0,n]})^S$, defined analogously to $S_k(CW_Q \mathcal{C}_{[0,n]})^\Gamma$ in Notation 10.5 but using the non-connective enrichment. The proof of Proposition 10.6 equally well shows that the forgetful functor $S_k(CW_Q \mathcal{C})^S \rightarrow S_k \mathcal{C}^S$ is a DK-equivalence. These non-connective enrichments are easier to understand because Proposition 2.11 implies that when we DK-embed $S_k \mathcal{C}^S$ in a pretriangulated spectral category, the DK-embedding takes cofiber sequences to distinguished triangles in the derived category. As a consequence, [4, 4.12] tells us that the maps

$$THH(S_k \mathcal{Q}^S) \longrightarrow THH(S_k(CW_Q \mathcal{C}_{[0,n-1]})^S) \longrightarrow THH(S_k(CW_Q \mathcal{C}_{[0,n]})^S)$$

are weak equivalences. Looking at the diagram

$$\begin{array}{ccc} |THH(S_\bullet(CW_Q \mathcal{C}_{[0,n-1]})^\Gamma)| & \longrightarrow & |THH(S_\bullet(CW_Q \mathcal{C}_{[0,n]})^\Gamma)| \\ \downarrow & & \downarrow \\ |THH(S_\bullet(CW_Q \mathcal{C}_{[0,n-1]})^S)| & \xrightarrow{\sim} & |THH(S_\bullet(CW_Q \mathcal{C}_{[0,n]})^S)|, \end{array}$$

we assume by induction on n that the lefthand map is a weak equivalence, the base case being the already known case of $S_\bullet CW_Q \mathcal{C}_{[0,0]} = S_\bullet \mathcal{Q}$. We then prove that the top map is a weak equivalence by showing that the righthand map is a weak equivalence.

To save space and eliminate unnecessary symbols, we will now write \mathcal{C}_k^n for $S_k(CW_Q \mathcal{C}_{[0,n]})$ or equivalently, $CW_{S_k \mathcal{Q}} S_k \mathcal{C}_{[0,n]}$, and $\mathcal{C}\Gamma_k^n$ and $\mathcal{C}S_k^n$ for the connective and non-connective spectral enrichments, respectively. Let $\mathcal{E}\Gamma_\bullet^n$ denote the simplicial spectral category where the objects of $\mathcal{E}\Gamma_k^n$ are the objects of \mathcal{C}_k^n and for objects X and Y , the mapping spectrum is the fiber product

$$\mathcal{E}\Gamma_k^n(X, Y) = S_k \mathcal{C}^\Gamma(x_{n-1}, y_{n-1}) \times_{S_k \mathcal{C}^\Gamma(x_{n-1}, y_n)} S_k \mathcal{C}^\Gamma(x_n, y_n).$$

We have a canonical simplicial spectral functor $\mathcal{E}\Gamma_\bullet^n \rightarrow \mathcal{C}\Gamma_\bullet^n$ sending X in $\mathcal{E}\Gamma_k^n$ to X viewed as an object of $\mathcal{C}\Gamma_k^n$ and using projection on the mapping spectra. We also have canonical simplicial spectral functors

$$\mathcal{E}\Gamma_\bullet^n \longrightarrow \mathcal{C}\Gamma_\bullet^{n-1} \quad \text{and} \quad \mathcal{E}\Gamma_\bullet^n \longrightarrow S_\bullet \Sigma^n \mathcal{Q}^\Gamma$$

sending X to its $(n-1)$ -skeleton X_{n-1} and to x_n/x_{n-1} , respectively, and performing the corresponding maps on mapping spectra. Using these maps, we can identify

$\mathcal{E}\Gamma_{\bullet}^n$ as the spectral category of extension sequences $[X_{n-1} \twoheadrightarrow X \twoheadrightarrow x_n/x_{n-1}]$ in \mathcal{C}_{\bullet}^n . Although the categories $\mathcal{C}\Gamma_{\bullet}^n$ and $\mathcal{E}\Gamma_{\bullet}^n$ do not exactly fit into the framework of Section 6, McCarthy's argument for the Additivity Theorem works quite generally and formally essentially using little more than the fact that the mapping spectra are functorial in the maps in S_{\bullet} ; the Additivity Theorem generalizes to the current context, and the argument following Corollary 6.2 shows that the maps described above induce a weak equivalence

$$|THH(\mathcal{E}\Gamma_{\bullet}^n)| \xrightarrow{\sim} |THH(\mathcal{C}\Gamma_{\bullet}^{n-1})| \times |THH(S_{\bullet}\Sigma^n \mathcal{Q}^{\Gamma})|.$$

We have an analogous simplicial spectral category $\mathcal{E}S_{\bullet}^n$ with the analogous weak equivalence. The induction hypothesis and the weak equivalences above then imply the following proposition.

Proposition 10.9. *The functor $\mathcal{E}\Gamma_{\bullet}^n \rightarrow \mathcal{E}S_{\bullet}^n$ induces a weak equivalence*

$$|THH(\mathcal{E}\Gamma_{\bullet}^n)| \xrightarrow{\sim} |THH(\mathcal{E}S_{\bullet}^n)|.$$

Let $(\mathcal{C}_k^n)^w$ denote the objects of \mathcal{C}_k^n that are weakly equivalent to the zero object $*$, and write $(\mathcal{C}\Gamma_k^n)^w$, $(\mathcal{C}S_k^n)^w$, $(\mathcal{E}\Gamma_k^n)^w$, $(\mathcal{E}S_k^n)^w$ for the various spectral enrichments on this category, the full spectral subcategories of $\mathcal{C}\Gamma_k^n$, $\mathcal{C}S_k^n$, $\mathcal{E}\Gamma_k^n$, $\mathcal{E}S_k^n$, respectively. The mapping spectra in $(\mathcal{C}\Gamma_k^n)^w$ and $(\mathcal{C}S_k^n)^w$ are all weakly contractible so THH is also weakly contractible,

$$THH((\mathcal{C}\Gamma_k^n)^w) \simeq THH((\mathcal{C}S_k^n)^w) \simeq *.$$

For the \mathcal{E} categories, we have the following proposition.

Proposition 10.10. *The canonical spectral functor $(\mathcal{E}\Gamma_k^n)^w \rightarrow (\mathcal{E}S_k^n)^w$ is a DK-equivalence.*

Proof. Since the categories have the same object set, it suffices to show that the map is a DK-embedding, and for this it suffices to show that the mapping spectra in $(\mathcal{E}S_k^n)^w$ are connective. We note that for X in $(\mathcal{C}_k^n)^w$, stability hypothesis (ii) implies that x_{n-1} is an object of $S_k \Sigma^{n-1} \mathcal{Q}$ since x_n is contractible and x_n/x_{n-1} is an object of $S_k \Sigma^n \mathcal{Q}$. Now given X and Y in $(\mathcal{C}_k^n)^w$, the projection map

$$\mathcal{E}S_k^n(X, Y) = S_k \mathcal{C}^S(x_{n-1}, y_{n-1}) \times_{S_k \mathcal{C}^S(x_{n-1}, y_n)} S_k \mathcal{C}^S(x_n, y_n) \rightarrow S_k \mathcal{C}^S(x_{n-1}, y_{n-1})$$

is a weak equivalence. In particular, $\mathcal{E}S_k^n(X, Y)$ is connective. \square

We denote by $CTHH(\mathcal{C}\Gamma_k^n, w)$ the homotopy cofiber of the inclusion

$$THH((\mathcal{C}\Gamma_k^n)^w) \rightarrow THH(\mathcal{C}\Gamma_k^n),$$

and analogously for $CTHH(\mathcal{C}S_k^n, w)$, $CTHH(\mathcal{E}\Gamma_k^n, w)$, and $CTHH(\mathcal{E}S_k^n, w)$. In this notation, the two previous propositions then imply the following proposition.

Proposition 10.11. *The map $|CTHH(\mathcal{E}\Gamma_{\bullet}^n, w)| \rightarrow |CTHH(\mathcal{E}S_{\bullet}^n, w)|$ is a weak equivalence.*

Since the inclusions

$$\begin{aligned} THH(\mathcal{C}\Gamma_k^n) &\rightarrow CTHH(\mathcal{C}\Gamma_k^n, w) \\ THH(\mathcal{C}S_k^n) &\rightarrow CTHH(\mathcal{C}S_k^n, w) \end{aligned}$$

are weak equivalences, the following lemma when combined with the previous proposition then completes the proof of Lemma 10.8.

Lemma 10.12. *For all k , the maps*

$$\begin{aligned} CTHH(\mathcal{E}\Gamma_k^n, w) &\longrightarrow CTHH(\mathcal{C}\Gamma_k^n, w) \\ CTHH(\mathcal{E}S_k^n, w) &\longrightarrow CTHH(\mathcal{C}S_k^n, w) \end{aligned}$$

are weak equivalences.

We prove the case for the connective enrichment in detail, the case for the non-connective enrichment being similar (but slightly easier). The statement is analogous to the Localization Theorem 6.1 of [4] except for the fact that the subcategories above are not pretriangulated. The proof goes roughly along the same lines as well, using the technology of THH with coefficients and the Dennis-Waldhausen Morita Argument of Section 5 of [4]; to keep this paper a reasonable length, we do not review this theory, but instead assume the reader is familiar with the ideas, notation, and terminology of Sections 5 and 6 of [4].

Define the $\mathcal{E}\Gamma_k^n$ -bimodule $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{C}\Gamma_k^n$ -bimodule $\mathcal{L}_{\mathcal{C}}$ by

$$\begin{aligned} \mathcal{L}_{\mathcal{E}}(X, Y) &= TB(\mathcal{E}\Gamma_k^n(-, Y); (\mathcal{E}\Gamma_k^n)^w; \mathcal{E}\Gamma_k^n(X, -)) \\ \mathcal{L}_{\mathcal{C}}(X, Y) &= TB(\mathcal{C}\Gamma_k^n(-, Y); (\mathcal{C}\Gamma_k^n)^w; \mathcal{C}\Gamma_k^n(X, -)). \end{aligned}$$

We then have maps of $\mathcal{E}\Gamma_k^n$ - and $\mathcal{C}\Gamma_k^n$ -bimodules

$$\mathcal{L}_{\mathcal{E}} \longrightarrow \mathcal{E}\Gamma_k^n \quad \text{and} \quad \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{C}\Gamma_k^n;$$

we let $\mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{C}}$ be the homotopy cofibers. Then as in [4, 6.4], the Dennis-Waldhausen Morita Argument gives us weak equivalences

$$\begin{aligned} THH(\mathcal{E}\Gamma_k^n; \mathcal{L}_{\mathcal{E}}) &\simeq THH((\mathcal{E}\Gamma_k^n)^w) & THH(\mathcal{E}\Gamma_k^n; \mathcal{M}_{\mathcal{E}}) &\simeq CTHH(\mathcal{E}\Gamma_k^n, w) \\ THH(\mathcal{C}\Gamma_k^n; \mathcal{L}_{\mathcal{C}}) &\simeq THH((\mathcal{C}\Gamma_k^n)^w) & THH(\mathcal{C}\Gamma_k^n; \mathcal{M}_{\mathcal{C}}) &\simeq CTHH(\mathcal{C}\Gamma_k^n, w), \end{aligned}$$

and we can identify the map in Lemma 10.12 as the map

$$(10.13) \quad THH(\mathcal{E}\Gamma_k^n; \mathcal{M}_{\mathcal{E}}) \longrightarrow THH(\mathcal{C}\Gamma_k^n; \mathcal{M}_{\mathcal{C}}).$$

As the mapping spectra in $(\mathcal{C}\Gamma_k^n)^w$ are weakly contractible, the spectra $\mathcal{L}_{\mathcal{C}}(X, Y)$ are weakly contractible for all X, Y , and it follows that the map of $\mathcal{C}\Gamma_k^n$ -bimodules $\mathcal{C}\Gamma_k^n \rightarrow \mathcal{M}_{\mathcal{C}}$ is a weak equivalence. We next move towards understanding the $\mathcal{E}\Gamma_k^n$ -bimodules $\mathcal{L}_{\mathcal{E}}$. We write u for the canonical functor $\mathcal{E}\Gamma_k^n \rightarrow \mathcal{C}\Gamma$ and also its restriction $(\mathcal{E}\Gamma_k^n)^w \rightarrow (\mathcal{C}\Gamma)^w$. We then have a commutative diagram of $\mathcal{E}\Gamma_k^n$ -bimodules

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{E}} & \longrightarrow & u^* \mathcal{L}_{\mathcal{C}} \\ \downarrow & & \downarrow \\ \mathcal{E}\Gamma_k^n & \longrightarrow & u^* \mathcal{C}\Gamma_k^n, \end{array}$$

Letting \mathcal{F} be the homotopy pullback of the deleted diagram

$$\mathcal{E}\Gamma_k^n \longrightarrow u^* \mathcal{C}\Gamma_k^n \longleftarrow u^* \mathcal{L}_{\mathcal{C}},$$

we get a map of $\mathcal{E}\Gamma_k^n$ -bimodules $\mathcal{L}_{\mathcal{E}} \rightarrow \mathcal{F}$.

Proposition 10.14. *The map of $\mathcal{E}\Gamma_k^n$ -bimodules $\mathcal{L}_{\mathcal{E}} \rightarrow \mathcal{F}$ is a weak equivalence.*

Proof. Fix X and Y objects in $\mathcal{E}\Gamma_k^n$; we need to show that the map $\mathcal{L}_{\mathcal{E}}(X, Y) \rightarrow \mathcal{F}(X, Y)$ is a weak equivalence. Consider the cofibration sequence

$$y_{n-1} \longrightarrow y_n \longrightarrow y_n/y_{n-1} \longrightarrow \Sigma y_{n-1}$$

obtained using a homotopy inverse weak equivalence to the collapse weak equivalence $y_n \cup_{y_{n-1}} Cy_{n-1} \rightarrow y_n/y_{n-1}$. By definition, y_n/y_{n-1} is in $S_k \Sigma^n \mathcal{Q}$, and since $n \geq 1$, there exists an object p in $S_k \Sigma^{n-1} \mathcal{Q}$ such that Σp is weakly equivalent to y_n/y_{n-1} . Then applying stability hypothesis (ii), we obtain from the cofibration sequence above a (homotopy class of) map $p \rightarrow y_{n-1}$ and a null homotopy $Cp \rightarrow y_n$ such that the induced map $\Sigma p \rightarrow y_n/y_{n-1}$ is homotopic to the chosen weak equivalence. Regarding Cp as an object of \mathcal{E}_k^n , it is an object of $(\mathcal{E}_k^n)^w$ and we have constructed a cellular map $Cp \rightarrow Y$. Consider the following commutative square.

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{E}}(X, Y) & \xleftarrow{a} & \mathcal{L}_{\mathcal{E}}(X, Cp) \\ \downarrow & & \downarrow c \\ \mathcal{F}(X, Y) & \xleftarrow{b} & \mathcal{F}(X, Cp) \end{array}$$

We complete the proof by arguing that the maps a , b , and c are weak equivalences.

To analyze the map a , consider an object Z in $(\mathcal{E}\Gamma_k^n)^w$. Since z_n is weakly equivalent to $*$ in \mathcal{C} , $\mathcal{E}\Gamma_k^n(Z, Y)$ is weakly equivalent to the homotopy fiber of the map $S_k \mathcal{C}^\Gamma(z_{n-1}, y_{n-1})$ to $S_k \mathcal{C}^\Gamma(z_{n-1}, y_n)$. We can use the cofibration sequence of Proposition 2.11 to understand this homotopy fiber: We have that $S_k \mathcal{C}^\Gamma(z_{n-1}, y_n/y_{n-1})$ is connected since z_{n-1} is an object of $S_k \Sigma^{n-1} \mathcal{Q}$ and y_n/y_{n-1} is an object of $S_k \Sigma^n \mathcal{Q}$. It follows that $\mathcal{E}\Gamma_k^n(Z, Y)$ is weakly equivalent to $\Omega S_k \mathcal{C}^\Gamma(z_{n-1}, y_n/y_{n-1})$. The same observations apply to Cp . Since by construction the map $\Sigma p = Cp/p \rightarrow y_n/y_{n-1}$ is a weak equivalence, we see by naturality that the map $\mathcal{E}\Gamma_k^n(Z, Cp) \rightarrow \mathcal{E}\Gamma_k^n(Z, Y)$ is a weak equivalence. Since this holds for any Z in $(\mathcal{E}\Gamma_k^n)^w$, unwinding the definition of $\mathcal{L}_{\mathcal{E}}$, we see that a is a weak equivalence.

For the map b , we note that $\mathcal{F}(X, Y)$ being the homotopy fiber of the map $\mathcal{E}\Gamma_k^n(X, Y)$ to $\mathcal{C}\Gamma(X, Y) = S_k \mathcal{C}^\Gamma(x_n, y_n)$, it is naturally weakly equivalent to the homotopy fiber of the map $S_k \mathcal{C}^\Gamma(x_{n-1}, y_{n-1})$ to $S_k \mathcal{C}^\Gamma(x_{n-1}, y_n)$. As in the previous case, we can identify this as $\Omega S_k \mathcal{C}^\Gamma(x_{n-1}, y_n/y_{n-1})$ since $S_k \mathcal{C}^\Gamma(x_{n-1}, y_n/y_{n-1})$ is connected (which can be proved by induction up the skeletal filtration of X using Proposition 2.11). Again, since the map $\Sigma p = Cp/p \rightarrow y_n/y_{n-1}$ is a weak equivalence, we see that b is a weak equivalence.

For the map c , since Cp is in $(\mathcal{E}\Gamma_k^n)^w$, the Two-Sided Bar Lemma [4, 5.3] implies that the natural map $\mathcal{L}_{\mathcal{E}}(X, Cp) \rightarrow \mathcal{E}\Gamma_k^n(X, Cp)$ is a weak equivalence. Since Cp is weakly equivalent to $*$ in \mathcal{C} , $\mathcal{C}\Gamma(X, Cp)$ is weakly contractible and we see that c is a weak equivalence. \square

The previous proposition lets us understand $\mathcal{M}_{\mathcal{E}}$.

Proposition 10.15. *The map of $\mathcal{E}\Gamma_k^n$ -bimodules $\mathcal{M}_{\mathcal{E}} \rightarrow u^* \mathcal{M}_{\mathcal{C}}$ is a weak equivalence.*

Proof. Since homotopy fiber squares in spectra are homotopy cocartesian, the canonical map from the homotopy cofiber of $\mathcal{F} \rightarrow \mathcal{E}\Gamma_k^n$ to the homotopy cofiber of $u^* \mathcal{L}_{\mathcal{C}} \rightarrow u^* \mathcal{C}\Gamma_k^n$ is a weak equivalence. \square

We now return to the map (10.13). We see from the previous proposition that we are in the situation where Proposition 5.6 of [4] applies. According to this proposition, to see that the map

$$THH(\mathcal{E}\Gamma_k^n; \mathcal{M}_{\mathcal{E}}) \longrightarrow THH(\mathcal{C}\Gamma_k^n; \mathcal{M}_{\mathcal{C}})$$

is a weak equivalence, we just need to check that the map

$$TB(\mathcal{C}\Gamma_k^n(-, Y); \mathcal{E}\Gamma_k^n; \mathcal{M}_C(X, -)) \longrightarrow \mathcal{C}\Gamma_k^n(X, Y)$$

is a weak equivalence for all X, Y in $\mathcal{C}\Gamma_k^n$, or equivalently in this case, for all X, Y in $\mathcal{E}\Gamma_k^n$. Since the Two-Sided Bar Lemma [4, 5.3] shows that the map

$$TB(\mathcal{C}\Gamma_k^n(-, Y); \mathcal{E}\Gamma_k^n; \mathcal{E}\Gamma_k^n(X, -)) \longrightarrow \mathcal{C}\Gamma_k^n(X, Y)$$

is a weak equivalence and $\mathcal{M}_C(X, -) \simeq \mathcal{M}_E(X, -)$ is the homotopy cofiber of $\mathcal{L}_E(X, -) \rightarrow \mathcal{E}\Gamma_k^n(X, -)$, it suffices to show that

$$\mathcal{G}(X, Y) = TB(\mathcal{C}\Gamma_k^n(-, Y); \mathcal{E}\Gamma_k^n; \mathcal{L}_E(X, -))$$

is weakly contractible. But we have

$$\begin{aligned} \mathcal{G}(X, Y) &= TB(\mathcal{C}\Gamma_k^n(-, Y); \mathcal{E}\Gamma_k^n; TB(\mathcal{E}\Gamma_k^n(-, -); (\mathcal{E}\Gamma_k^n)^w; \mathcal{E}\Gamma_k^n(X, -))) \\ &\simeq TB(\mathcal{C}\Gamma_k^n(-, Y); (\mathcal{E}\Gamma_k^n)^w; \mathcal{E}\Gamma_k^n(X, -)). \end{aligned}$$

Since $\mathcal{C}\Gamma_k^n(Z, Y)$ is weakly contractible for any Z in $(\mathcal{E}\Gamma_k^n)^w$, it follows that $\mathcal{G}(X, Y)$ is weakly contractible. This completes the proof that (10.13) is a weak equivalence and hence the proof of Lemma 10.12, which in turn completes the proof of Lemma 10.8.

Part 3. The localization sequence for $THH(ku)$

11. THE LOCALIZATION SEQUENCE FOR $THH(ku)$

The work in this paper is motivated in large part by the need to develop suitable machinery to prove the following localization theorem, conjectured by Hesselholt and Ausoni-Rognes [3, 1, 2]. This generalizes work of Hesselholt and Madsen at chromatic level 0 [13].

Theorem 11.1. *The transfer maps and the canonical maps fit into cofiber sequences of cyclotomic spectra*

$$\begin{aligned} THH(\mathbb{Z}_p^\wedge) &\longrightarrow THH(\ell_p^\wedge) \longrightarrow WTHH^\Gamma(\ell_p^\wedge | L_p^\wedge) \longrightarrow \Sigma THH(\mathbb{Z}_p^\wedge) \\ THH(\mathbb{Z}_{(p)}) &\longrightarrow THH(\ell) \longrightarrow WTHH^\Gamma(\ell | L) \longrightarrow \Sigma THH(\mathbb{Z}_{(p)}) \\ THH(\mathbb{Z}) &\longrightarrow THH(ku) \longrightarrow WTHH^\Gamma(ku | KU) \longrightarrow \Sigma THH(\mathbb{Z}) \end{aligned}$$

inducing cofiber sequences

$$\begin{aligned} TC(\mathbb{Z}_p^\wedge) &\longrightarrow TC(\ell_p^\wedge) \longrightarrow WTC^\Gamma(\ell_p^\wedge | L_p^\wedge) \longrightarrow \Sigma TC(\mathbb{Z}_p^\wedge) \\ TC(\mathbb{Z}_{(p)}) &\longrightarrow TC(\ell) \longrightarrow WTC^\Gamma(\ell | L) \longrightarrow \Sigma TC(\mathbb{Z}_{(p)}) \\ TC(\mathbb{Z}) &\longrightarrow TC(ku) \longrightarrow WTC^\Gamma(ku | KU) \longrightarrow \Sigma TC(\mathbb{Z}) \end{aligned}$$

which are compatible via the cyclotomic trace with the corresponding cofiber sequences in algebraic K -theory constructed in [5].

Here $WTHH^\Gamma(ku | KU)$ denotes the connective THH of the category of finite cell ku -modules with the spectral enrichment induced by the canonical mapping spaces in ku but weak equivalences the KU -equivalences. That is,

$$WTHH^\Gamma(ku | KU) = WTHH^\Gamma(\mathcal{C}_{ku} | v) = \Omega | THH(v_\bullet^M S_\bullet \mathcal{C}_{ku}^\Gamma) |,$$

where \mathcal{C}_{ku} is the category of finite cell EKMM ku -modules (as in Example 1.5.(i)) and $v\mathcal{C}_{ku}$ is the collection of maps $M \rightarrow N$ such that $M \wedge_{ku} KU \rightarrow N \wedge_{ku} KU$ is an

equivalence, or equivalently, those maps that induce an isomorphism on homotopy groups after inverting the action of the Bott element.

The proof of this theorem follows the same general outline as the proof of the corresponding result in algebraic K -theory [5]. In particular, the localization theorem follows from a “dévissage” theorem for finitely generated finite stage Postnikov towers. We now give the definitions necessary to state this theorem. Throughout, we work with EKMM S -algebras and S -modules.

For an S -algebra R , let \mathcal{P}_R denote the full subcategory of left R -modules that are of the homotopy type of cell R -modules and have only finitely many non-zero homotopy groups, all of which are finitely generated over $\pi_0 R$. We give \mathcal{P}_R the structure of a simplicially tensored Waldhausen category as follows. For the simplicial structure, we use the usual simplicial enrichment obtained by regarding the category of R -modules as a simplicial model category. For the Waldhausen category structure, we take the weak equivalences to be the usual weak equivalences and the cofibrations to be the Hurewicz cofibrations, i.e., the maps satisfying the homotopy extension property in the category of R -modules. As we described in [5, §1], this gives \mathcal{P}_R the structure of a Waldhausen category, and the pushout-product axiom on the tensors follows from [11, X.2.3]. (Techniques to make a version of \mathcal{P}_R that is a small category are discussed in [5, 1.7].)

Restricting to the subcategory of the category of S -algebras with morphisms the maps $R \rightarrow R'$ for which $\pi_0 R'$ is finitely generated as a left $\pi_0 R$ -module, we can regard $WTHH^\Gamma(\mathcal{P}_{(-)})$ as a contravariant functor to the homotopy category of cyclotomic spectra. We can now state the Dévissage Theorem.

Theorem 11.2 (Dévissage Theorem). *Let R be a connective S -algebra with $\pi_0 R$ left Noetherian. Then there is a natural isomorphism in the homotopy category of cyclotomic spectra $THH(\mathcal{E}_{\pi_0 R}^{fg}) \rightarrow WTHH^\Gamma(\mathcal{P}_R)$, where $\mathcal{E}_{\pi_0 R}^{fg}$ denotes the exact category of finitely generated left $\pi_0 R$ -modules. Moreover, this isomorphism and the induced isomorphism (in the stable category) on TC are compatible via the cyclotomic trace with the analogous isomorphism (in the stable category) on algebraic K -theory $K'(\pi_0 R) \rightarrow K'(R)$ in the Dévissage Theorem of [5].*

We prove Theorem 11.2 in the next section and use the rest of this section to prove Theorem 11.1 from Theorem 11.2. Let R be one of ku , ℓ , or ℓ_p^\wedge , and let β denote the appropriate Bott element in $\pi_* R$ in degree 2 or $2p - 2$. Then $R[\beta^{-1}]$ is KU , L , or L_p^\wedge respectively. For convenience, let Z denote $\pi_0 R$; so $Z = \mathbb{Z}$, $\mathbb{Z}_{(p)}$, or \mathbb{Z}_p^\wedge in the respective cases. As above we write \mathcal{C}_A for the simplicially tensored Waldhausen category of finite cell A -modules (where $A = HZ$, R , or $R[\beta^{-1}]$). On \mathcal{C}_R we have the additional weak equivalences $v\mathcal{C}_R$, the maps that induce an isomorphism on homotopy groups after inverting the action of the Bott element. Since $v\mathcal{C}_R$ contains the usual weak equivalences $w\mathcal{C}_R$, the hypothesis of the Localization Theorem (Theorem 8.1) applies and we get a cofibration sequence of cyclotomic spectra

$$WTHH^\Gamma(\mathcal{C}_R^v) \longrightarrow WTHH^\Gamma(\mathcal{C}_R) \longrightarrow WTHH^\Gamma(\mathcal{C}_R|v) \longrightarrow \Sigma WTHH^\Gamma(\mathcal{C}_R^v),$$

compatible with the analogous sequence in K -theory via the cyclotomic trace. Corollary 9.4 identifies $WTHH^\Gamma(\mathcal{C}_R)$ with $THH(R)$, compatibly with the cyclotomic trace. The inclusion of the v -acyclics \mathcal{C}_R^v into the simplicially tensored

Waldhausen category \mathcal{P}_R described above is a tensored exact functor and a DK-equivalence. Thus, Theorem 11.2 identifies \mathcal{C}_R^v as $THH(Z)$, compatibly with the identification of $K(\mathcal{C}_R^v)$ with $K(HZ)$.

This completes most of the proof of Theorem 11.1; it just remains to identify the map

$$THH(Z) \simeq WTHH^\Gamma(\mathcal{C}_R^v) \longrightarrow WTHH^\Gamma(\mathcal{C}_R) \simeq THH(R)$$

in terms of the transfer map $THH(HZ) \rightarrow THH(R)$. First, we review this transfer map. In our current context with $R = ku, \ell$, or ℓ_p^\wedge , the Eilenberg-Mac Lane R -module HZ is weakly equivalent to a finite cell R -module. If we choose a model for HZ as a cofibrant associative R -algebra, then finite cell HZ -modules are cell R -modules and homotopy equivalent to finite cell R -modules. Let \mathcal{M}_R^c be the simplicially tensored Waldhausen category whose objects are the R -modules that are homotopy equivalent to finite cell R -modules with the usual simplicial sets of maps, with the usual weak equivalences, and with cofibrations the Hurewicz cofibrations (using the technique of [5, 1.7] to make a version that is a small category). Then \mathcal{P}_R is a closed Waldhausen subcategory of \mathcal{M}_R^c ; moreover, the inclusion of \mathcal{C}_R in \mathcal{M}_R^c is tensored exact and a DK-equivalence, and so induces an equivalence on all versions of THH . We also have the analogous category \mathcal{M}_{HZ}^c for HZ , which coincides with \mathcal{P}_{HZ} . The forgetful functor from HZ -modules to R -modules is a tensored exact functor $\mathcal{M}_{HZ}^c \rightarrow \mathcal{M}_R^c$. The transfer map $THH(HZ) \rightarrow THH(R)$ is by definition the map

$$\tau_{HZ}^R: THH(HZ) \longrightarrow THH(\mathcal{M}_R^c) \xleftarrow{\sim} THH(R),$$

induced by the inclusion of S_{HZ} in \mathcal{M}_R^c and the induced map of endomorphism spectra

$$\mathcal{C}_{HZ}^S(S_{HZ}, S_{HZ}) = (\mathcal{M}_{HZ}^c)^S(S_{HZ}, S_{HZ}) \longrightarrow (\mathcal{M}_R^c)^S(S_{HZ}, S_{HZ}).$$

(We understand THH of the EKMM S -algebra HZ as THH of the symmetric ring spectrum $\mathcal{C}_{HZ}^S(S_{HZ}, S_{HZ})$; cf. Corollary 9.4 and the remarks that follow it.)

Since the transfer map coincides with the map

$$THH(HZ) \xrightarrow{\sim} THH(\mathcal{M}_{HZ}^c) \longrightarrow THH(\mathcal{M}_R^c) \xleftarrow{\sim} THH(R),$$

applying Corollary 9.4 and naturality, we can also identify it as the map

$$THH(HZ) \simeq WTHH^\Gamma(\mathcal{M}_{HZ}^c) \longrightarrow WTHH^\Gamma(\mathcal{M}_R^c) \simeq THH(R).$$

Using the naturality of the isomorphism in Theorem 11.2, we obtain the following commutative diagram of maps in the homotopy category of cyclotomic spectra.

$$\begin{array}{ccccccc} THH(Z) & \xrightarrow{\sim} & WTHH^\Gamma(\mathcal{P}_{HZ}) & \xrightarrow{=} & WTHH^\Gamma(\mathcal{M}_{HZ}^c) & \xleftarrow{\sim} & THH(HZ) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow \tau_{HZ}^R \\ THH(Z) & \xrightarrow{\sim} & WTHH^\Gamma(\mathcal{P}_R) & \longrightarrow & WTHH^\Gamma(\mathcal{M}_R^c) & \xleftarrow{\sim} & THH(R) \\ & & \uparrow \sim & & \uparrow \sim & \nearrow \sim & \\ & & WTHH^\Gamma(\mathcal{C}_R^v) & \longrightarrow & WTHH^\Gamma(\mathcal{C}_R) & & \end{array}$$

It will be obvious from the proof of Theorem 11.2 in the next section that the isomorphism $THH(Z) \simeq THH(HZ)$ in the top row of the diagram is the standard

one, and this identifies the map $THH(Z) \rightarrow THH(R)$ as the transfer map. This completes the proof of Theorem 11.1.

12. PROOF OF THE DÉVISSAGE THEOREM

This section is devoted to the proof of the Dévissage Theorem, Theorem 11.2. The argument parallels the analogous dévissage theorem in [5], which we review along the way.

We fix the connective S -algebra R , writing \mathcal{P} for \mathcal{P}_R . Let \mathcal{P}_m^n denote the full subcategory of \mathcal{P} consisting of those R -modules whose homotopy groups π_q are zero for $q > n$ or $q < m$. In this notation, we permit $m = -\infty$ and/or $n = \infty$, so $\mathcal{P} = \mathcal{P}_{-\infty}^\infty$. The categories \mathcal{P}_m^n are closed Waldhausen subcategories of \mathcal{P}_R . The following theorem proved below parallels [5, 1.2].

Theorem 12.1. *The inclusion $\mathcal{P}_0^0 \rightarrow \mathcal{P}$ induces a weak equivalence*

$$WTHH^\Gamma(\mathcal{P}_0^0) \longrightarrow WTHH^\Gamma(\mathcal{P}).$$

The point of the previous theorem is that π_0 provides an exact functor from \mathcal{P}_0^0 to the exact category of finitely generated left $\pi_0 R$ -modules \mathcal{E}_R^{fg} . Theorem 1.3 of [5] proves that this functor induces a weak equivalence of K -theory. Since the simplicial mapping sets for \mathcal{E}_R^{fg} are discrete, π_0 is also a simplicially enriched functor $\mathcal{P}_0^0 \rightarrow \mathcal{E}_R^{fg}$. It is in fact a DK-equivalence and induces a DK-equivalence $S_n \mathcal{P}_0^0 \rightarrow S_n \mathcal{E}_R^{fg}$ for all n . This proves the following theorem, which parallels [5, 1.3].

Theorem 12.2. *The functor $\pi_0: \mathcal{P}_0^0 \rightarrow \mathcal{E}_R^{fg}$ induces a weak equivalence*

$$WTHH^\Gamma(\mathcal{P}_0^0) \longrightarrow WTHH^\Gamma(\mathcal{E}_R^{fg}) = THH(\mathcal{E}_R^{fg}).$$

Theorem 11.2 is an immediate consequence of the previous two theorems, with the natural isomorphism coming from the natural zigzag of weak equivalences of cyclotomic spectra

$$THH(\mathcal{E}_R^{fg}) = WTHH^\Gamma(\mathcal{E}_R^{fg}) \longleftarrow WTHH^\Gamma(\mathcal{P}_0^0) \longrightarrow WTHH^\Gamma(\mathcal{P}).$$

Thus, it remains to prove Theorem 12.1.

The proof of Theorem 12.1 follows the same outline as the parallel theorem [5, 1.2]. As in the argument there, we have the following two easy observations.

Proposition 12.3. *The inclusion $\mathcal{P}_0^\infty \rightarrow \mathcal{P}$ induces an equivalence*

$$WTHH^\Gamma(\mathcal{P}_0^\infty) \longrightarrow WTHH^\Gamma(\mathcal{P}).$$

Proposition 12.4. *The cyclotomic spectrum $WTHH^\Gamma(\mathcal{P}_0^\infty)$ is weakly equivalent to the telescope of the sequence of maps*

$$WTHH^\Gamma(\mathcal{P}_0^0) \longrightarrow \cdots \longrightarrow WTHH^\Gamma(\mathcal{P}_0^n) \longrightarrow WTHH^\Gamma(\mathcal{P}_0^{n+1}) \longrightarrow \cdots.$$

As in [5], the proof of Theorem 12.1 will then be completed by showing that the maps

$$WTHH^\Gamma(\mathcal{P}_0^n) \longrightarrow WTHH^\Gamma(\mathcal{P}_0^{n+1})$$

are weak equivalences for all $n \geq 0$. Applying Proposition 7.6 and Theorem 7.2, this is equivalent to proving the following lemma.

Lemma 12.5. *$WTHH^\Gamma(\mathcal{P}_0^n \rightarrow \mathcal{P}_0^{n+1}) \simeq \Omega||THH(S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma)||$ is weakly contractible.*

In [5] the proof of the parallel (unnumbered) lemma consisted of several steps, each of which compared (multi)simplicial sets; the following diagram outlines the comparisons as stated there.

$$\begin{array}{ccccc}
 w_{\bullet} S_{\bullet} F_{\bullet}(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) & & u_{\bullet} S_{\bullet} \mathcal{P}_0^{n+1} & \longrightarrow & u_{\bullet} S'_{\bullet} \mathcal{P}_0^{n+1} \cdots \longrightarrow & u_{\bullet} M_{\bullet} Z \\
 & \searrow & \swarrow & & \downarrow & \nearrow \\
 & & w_{\bullet} u_{\bullet} S_{\bullet} \mathcal{P}_0^{n+1} & & u_{\bullet} F_{\bullet-1}^f \mathcal{P}_0^{n+1} \longleftarrow \cdots & u_{\bullet} F_{\bullet-1}^f \mathcal{P}_{n+1}^{n+1} \\
 & & & & \uparrow & \downarrow
 \end{array}$$

We review these constructions as needed below. Here the solid arrows are simplicial maps of diagonal simplicial sets and the dotted arrows are maps that are simplicial only in one of the simplicial directions. We correct a minor error in [5] below. There we claimed that the dotted arrow in the top row was a map of bisimplicial sets; it is not. The diagram for the corrected argument looks like this; it commutes up to simplicial homotopy.

$$\begin{array}{ccccccc}
 w_{\bullet} S_{\bullet} F_{\bullet}(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) & & u_{\bullet} S_{\bullet} \mathcal{P}_0^{n+1} & \longrightarrow & u_{\bullet} S'_{\bullet} \mathcal{P}_0^{n+1} & \longrightarrow & u_{\bullet} S^f M_{\bullet} Z \longleftarrow u_{\bullet} M_{\bullet} Z \\
 & \searrow & \swarrow & & \downarrow & & \nearrow \\
 & & w_{\bullet} u_{\bullet} S_{\bullet} \mathcal{P}_0^{n+1} & & u_{\bullet} F_{\bullet-1}^f \mathcal{P}_0^{n+1} & \longleftarrow \cdots & u_{\bullet} F_{\bullet-1}^f \mathcal{P}_{n+1}^{n+1} \\
 & & & & \uparrow & & \downarrow
 \end{array}$$

In the current context of THH , the line of reasoning and the diagram simplifies slightly; we use the following diagram of spectrally enriched functors, which commutes up to natural isomorphism.

$$\begin{array}{ccccccc}
 (S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n))^{\Gamma} & \xrightarrow{\sim} & ((u_q S'_p)^M \mathcal{P}_0^{n+1})^{\Gamma} & \longrightarrow & (u_q S^f M_p)^{\Gamma} & \xleftarrow{\sim} & (u_q M_p Z)^{\Gamma} \\
 & & \downarrow \sim & & \nearrow & & \uparrow \sim \\
 (12.6) & & ((u_q F_{p-1}^f)^M \mathcal{P}_0^{n+1})^{\Gamma} & \longleftarrow \cdots & & & ((u_q F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^{\Gamma}
 \end{array}$$

All of the spectral categories fit into simplicial spectral categories (in the q direction) and the ones on the top row fit into bisimplicial spectral categories (in p, q). The solid arrows are the spectrally enriched functors that respect the bisimplicial structure; the dotted arrows respect the simplicial structure in the q direction. The arrows marked “ \sim ” are DK-equivalences, as shown in Propositions 12.8, 12.9, 12.13, and 12.16. The goal is to show that the composite functor $(S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n))^{\Gamma} \rightarrow (u_q S^f M_p Z)^{\Gamma}$ induces a weak equivalence

$$||THH((S_{\bullet} F_{\bullet}(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n))^{\Gamma})|| \longrightarrow ||THH((u_{\bullet} S^f M_{\bullet} Z)^{\Gamma})|| \simeq ||THH((u_{\bullet} M_{\bullet} Z)^{\Gamma})||$$

and then prove Lemma 12.5 by showing that $||THH((u_{\bullet} M_{\bullet} Z)^{\Gamma})||$ is contractible (Proposition 12.12).

We now begin to review the categories and maps in diagram (12.6). We use the following notation.

Definition 12.7. Let $u\mathcal{P}$ denote the subcategory of \mathcal{P} consisting of those maps that induce an isomorphism on π_{n+1} and an injection on π_n . Let $f\mathcal{P}$ denote the subcategory of \mathcal{P} consisting of those maps that induce an epimorphism on π_0 .

We write $u_{\bullet}\mathcal{P}$ for the nerve categories: An object of $u_q\mathcal{P}$ is a sequence of q composable maps in $u\mathcal{P}$ and a map in $u_q\mathcal{P}$ is a commuting diagram (of maps in \mathcal{P}).

For consistency with [5, 3.7], we let $F_p^f \mathcal{P}$ denote the nerve category $f_p \mathcal{P}$: An object is a sequence of p composable maps in $f \mathcal{P}$ and a map is a commuting diagram (of maps in \mathcal{P}). We extend the definition of u_\bullet in the obvious way to functor categories: In diagram (12.6), the category $u_q S_p \mathcal{P}_0^{n+1}$ has as objects the sequences of q -composable maps

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_q} A_q$$

between objects A_i in $S_p \mathcal{P}_0^{n+1}$ where each α_i is (objectwise) in $u \mathcal{P}_0^{n+1}$; a map from $\{\alpha_i\}$ to $\{\alpha'_i\}$ consists of a map $\phi_i: A_i \rightarrow A'_i$ in $S'_p \mathcal{P}_0^{n+1}$ for each i , making the diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_q} & A_q \\ \phi_0 \downarrow & & \phi_1 \downarrow & & & & \downarrow \phi_p \\ A'_0 & \xrightarrow{\alpha'_1} & A'_1 & \xrightarrow{\alpha'_2} & \cdots & \xrightarrow{\alpha'_q} & A'_q \end{array}$$

in $S'_p \mathcal{P}_0^{n+1}$ commute. We define the categories $u_q F_{p-1}^f \mathcal{P}_0^{n+1}$ and $u_q F_{p-1}^f \mathcal{P}_{n+1}^{n+1}$ analogously in terms of composable maps and diagrams in $F_{p-1}^f \mathcal{P}_0^{n+1}$ and $F_{p-1}^f \mathcal{P}_{n+1}^{n+1}$. We obtain the spectrally enriched categories $((u_q S'_p)^M \mathcal{P}_0^{n+1})^\Gamma$, $((u_q F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$, and $((u_q F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma$ using the Moore Tot mapping spaces (Construction 4.4) and the connective spectral enrichment. The usual face and degeneracy maps in the nerve construction makes $((u_\bullet S'_\bullet)^M \mathcal{P}_0^{n+1})^\Gamma$ into a bisimplicial spectral category and make $((u_\bullet F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$ and $((u_\bullet F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma$ into simplicial spectral categories for each $p > 0$.

Next we review the canonical inclusion

$$F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) \longrightarrow u_q \mathcal{P}_0^{n+1}.$$

We recall that an object of $F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$ consists of a sequence of q composable cofibrations in \mathcal{P}_0^{n+1}

$$x_0 \rightrightarrows x_1 \rightrightarrows \cdots \rightrightarrows x_q$$

such that each quotient x_{i+1}/x_i is in \mathcal{P}_0^{n+1} . We note that for a cofibration $j: a \rightarrow b$ in \mathcal{P} between objects of \mathcal{P}_0^{n+1} , the quotient b/a is in \mathcal{P}_0^{n+1} if and only if j induces an isomorphism on π_{n+1} and an injection on π_n , that is, if and only if j is in $u \mathcal{P}$. It follows that $F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$ is the full subcategory of $u_q \mathcal{P}_0^{n+1}$ consisting of those objects whose structure maps are cofibrations. We then obtain the functors

$$S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) \longrightarrow u_q S'_p \mathcal{P}_0^{n+1}$$

as the corresponding inclusions of full subcategories. When we look at mapping spaces and use the Moore enrichment, we obtain a DK-embedding

$$S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) \longrightarrow (u_q S'_p)^M \mathcal{P}_0^{n+1}.$$

This map is a DK-equivalence since the usual cylinder argument replacing a map with a cofibration converts any diagram in $u_q S'_p \mathcal{P}_0^{n+1}$ to a weakly equivalent diagram in $S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$. Passing to the connective spectral enrichments, we obtain the following proposition.

Proposition 12.8. *The spectrally enriched functor*

$$S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma \longrightarrow ((u_q S'_p)^M \mathcal{P}_0^{n+1})^\Gamma$$

is a *DK-equivalence*.

Next we review the functor $S'_p \mathcal{P}_0^{n+1} \rightarrow F_{p-1}^f \mathcal{P}_0^{n+1}$ of [5, 3.8]. First note that for an object $A = \{a_{i,j}\}$ in $S'_p \mathcal{P}_0^{n+1}$, the map $a_{i,p} \rightarrow a_{j,p}$ is the cofiber of the map $a_{i,j} \rightarrow a_{i,p}$ and so we have a long exact sequence of homotopy groups

$$0 \longrightarrow \pi_{n+1} a_{i,j} \longrightarrow \cdots \longrightarrow \pi_0 a_{i,j} \longrightarrow \pi_0 a_{i,p} \longrightarrow \pi_0 a_{j,p} \longrightarrow 0.$$

In particular, the map $a_{i,p} \rightarrow a_{j,p}$ is surjective on π_0 , that is, is a map in $f\mathcal{P}_0^{n+1}$. We therefore obtain a functor $S'_p \mathcal{P}_0^{n+1} \rightarrow F_{p-1}^f \mathcal{P}_0^{n+1}$ by sending each object of $S'_p \mathcal{P}_0^{n+1}$ to the object of $F_{p-1}^f \mathcal{P}_0^{n+1}$ defined by the sequence

$$a_{0,p} \longrightarrow a_{2,p} \longrightarrow \cdots \longrightarrow a_{p-1,p}.$$

In fact we have the following proposition.

Proposition 12.9. *The spectrally enriched functor*

$$((u_q S'_p)^M \mathcal{P}_0^{n+1})^\Gamma \longrightarrow ((u_q F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$$

is a DK-equivalence.

Proof. Although $S'_p \mathcal{P}_0^{n+1}$ is defined in terms of homotopy cocartesian squares, it could equally well be defined in terms of homotopy cartesian squares since for EKMM R -modules a square is homotopy cartesian if and only if it is homotopy cocartesian. The description of the mapping space of $S_p \mathcal{P}$ in (3.2) has an analogue in this context: The canonical map from $S_p^M \mathcal{P}$ to the iterated homotopy pullback

$$\mathcal{P}(a_{0,p}, b_{0,p}) \times_{\mathcal{P}(a_{0,p}, b_{1,p})}^h \cdots \times_{\mathcal{P}(a_{p-2,p}, b_{p-1,p})}^h \mathcal{P}(a_{p-1,p}, b_{p-1,p}).$$

is a weak equivalence. This extends to $(u_q S'_p)^M \mathcal{P}$ and from this it is easy to deduce that we have a DK-embedding. It is a DK-equivalence because every object of $u_q F_{p-1}^f \mathcal{P}_0^{n+1}$ is weakly equivalent to the image of an object in $u_q S'_p \mathcal{P}_0^{n+1}$, filling out the diagram by taking homotopy fibers. \square

The inclusion of \mathcal{P}_{n+1}^{n+1} as a subcategory of \mathcal{P}_0^{n+1} induces a spectrally enriched functor $((u_q F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \rightarrow ((u_q F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$, which assembles to a simplicial spectrally enriched functor in the q direction. Although not a DK-equivalence at any level, the simplicial spectrally enriched functor does induce a weak equivalence on THH .

Proposition 12.10. *The inclusion $((u_\bullet F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \rightarrow ((u_\bullet F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$ induces a weak equivalence*

$$|THH(((u_\bullet F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma)| \longrightarrow |THH(((u_\bullet F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma)|.$$

Proof. Consider the bisimplicial spectral category $V_{\bullet\bullet}^\Gamma$ defined as follows: in bidegree r, s , $V_{r,s}^\Gamma$ is the full spectral subcategory of $((u_{r+s+1} F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$ with objects the sequences of sequences of the form

$$a_0 \longrightarrow \cdots \longrightarrow a_r \longrightarrow b_0 \longrightarrow \cdots \longrightarrow b_s$$

such that the objects a_i are in $F_{p-1}^f \mathcal{P}_{n+1}^{n+1}$. Dropping the objects $\{a_i\}$ and the objects $\{b_i\}$ respectively induce bisimplicial spectrally enriched functors

$$((u_r F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \longleftarrow V_{r,s}^\Gamma \longrightarrow ((u_s F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma,$$

where we regard the targets as constant bisimplicial objects in the appropriate direction. Since the (connective) spectrum of maps from an object x of \mathcal{P}_{n+1}^{n+1} to an object y of \mathcal{P}_0^{n+1} is homotopy discrete with $\pi_0 = \text{Hom}_{\mathbb{Z}}(\pi_{n+1}x, \pi_{n+1}y)$, we see that the map $V_{r,s}^\Gamma \rightarrow ((u_s F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$ is a DK-embedding. Furthermore, it is clear that this functor is essentially surjective (choosing an n -connected cover of b_0), and so is a DK-equivalence.

The usual arguments show that the map $V_{r,s}^\Gamma \rightarrow ((u_r F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma$ is a simplicial homotopy equivalence in the s -direction, using the homotopy inverse induced by

$$(a_0 \longrightarrow \cdots \longrightarrow a_r) \quad \mapsto \quad (a_0 \longrightarrow \cdots \longrightarrow a_r = a_r = \cdots = a_r).$$

Using this homotopy inverse, the composite map on (diagonal) simplicial spectral categories

$$((u_r F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \longrightarrow V_{r,r}^\Gamma \longrightarrow ((u_r F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma$$

is induced by

$$(a_0 \longrightarrow \cdots \longrightarrow a_r) \quad \mapsto \quad (a_r = a_r = \cdots = a_r),$$

and is easily seen to be simplicially homotopic to the inclusion map. \square

For the categories $u_q M_p Z$, we copy the following definition from [5, 3.9].

Definition 12.11. Let $Z = \pi_0 R$. Let $M_p Z$ be the category whose objects are sequences of $p - 1$ composable maps of finitely generated left Z -modules $x_0 \rightarrow \cdots \rightarrow x_{p-1}$ and whose morphisms are commutative diagrams. Let $u M_p Z$ be the subcategory of $M_p Z$ consisting of all objects but only those maps $x \rightarrow y$ that are isomorphisms $x_i \rightarrow y_i$ for all $0 \leq i \leq p - 1$.

We understand $M_0 Z$ to be the trivial category consisting of a single object (the empty sequence of maps) with only the identity map. As above, we let $u_q M_p Z$ denote the nerve category, which has as its objects the composable sequences of q maps in $u M_p Z$ (i.e., isomorphisms in $M_p Z$) and maps the commutative diagrams of maps in $M_p Z$. We regard $u_q M_p Z$ as simplicially enriched with discrete mapping spaces and we obtain a connective spectral enrichment $(u_q M_p Z)^\Gamma$ using objectwise direct sum of finitely generated left Z -modules.

As above, $(u_\bullet M_p Z)^\Gamma$ assembles into a simplicial spectral category using the usual face and degeneracy maps for the nerve. We make $(u_\bullet M_\bullet Z)^\Gamma$ into a bisimplicial spectral category as follows: For $0 \leq i \leq p - 1$, on $x_0 \rightarrow \cdots \rightarrow x_{p-1}$, the face map $\partial_i : u_q M_p Z \rightarrow u_q M_{p-1} Z$ is defined by dropping x_i (and composing) and the degeneracy map $s_i : M_{p-1} Z \rightarrow M_p Z$ is defined by repeating x_i (with the identity map). The face map $\partial_p : M_p Z \rightarrow M_{p-1} Z$ sends $x_0 \rightarrow \cdots \rightarrow x_{p-1}$ to $k_0 \rightarrow \cdots \rightarrow k_{p-2}$, where $k_i \subset x_i$ is the kernel of the composite map $x_i \rightarrow x_{p-1}$. The last degeneracy $s_{p-1} : M_{p-1} Z \rightarrow M_p Z$ puts 0 in as the last object in the sequence. The fundamental property of $(u_\bullet M_\bullet Z)^\Gamma$ that we need is the following.

Proposition 12.12. *For each q , $|THH((u_q M_\bullet Z)^\Gamma)|$ is contractible.*

Proof. The argument at the end of Section 3 of [5] constructs a simplicial contraction on the simplicial spectral category $(u_q M_\bullet Z)^\Gamma$. This simplicial contraction induces a simplicial contraction on the simplicial spectrum $THH((u_q M_\bullet Z)^\Gamma)$ and geometric realization converts this to a contraction of $|THH((u_q M_\bullet Z)^\Gamma)|$. \square

Applying π_{n+1} , we get a functor $uF_{p-1}^f \mathcal{P}_0^{n+1} \rightarrow uM_{p-1}$ and spectrally enriched functors

$$((u_q F_{p-1}^f)^M \mathcal{P}_0^{n+1})^\Gamma \longrightarrow (u_q M_p Z)^\Gamma \quad \text{and} \quad ((u_q F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \longrightarrow (u_q M_p Z)^\Gamma.$$

Looking at the mapping spaces and mapping spectra, the following proposition is clear.

Proposition 12.13. *The spectrally enriched functor*

$$((u_q F_{p-1}^f)^M \mathcal{P}_{n+1}^{n+1})^\Gamma \longrightarrow (u_q M_p Z)^\Gamma$$

is a DK-equivalence.

In [5, §3], we claimed that the functors $uS'_p \mathcal{P}_0^{n+1} \rightarrow uM_p Z$ respected the simplicial structure in the p direction, which is untrue. To fix this, we introduce the category $uS^f M_\bullet Z$.

Definition 12.14. Let $S^f M_p Z$ be the category whose objects are functors $A = a_{-, -}$ from $\text{Ar}[p]$ to the category of finitely generated left Z -modules such that:

- (i) $a_{i,i} = 0$, and
- (ii) $a_{i,j} \rightarrow a_{i,k}$ is an isomorphism onto the kernel of the map $a_{i,k} \rightarrow a_{j,k}$

for all $i \leq j \leq k$. A map in $S^f M_p Z$ is a commutative diagram. The subcategory $uS^f M_p Z$ consists of those maps in $S^f M_p$ that are isomorphisms.

We make $uS^f M_\bullet Z$ a simplicial category using the usual face and degeneracy operations on $\text{Ar}[\bullet]$. Basically $S^f M_\bullet Z$ is the fibration version of the S_\bullet construction for the co-Waldhausen category (category with *fibrations* and weak equivalences) structure we get on the category of finitely generated left Z -modules by taking the fibrations to be all maps and the weak equivalences to be the isomorphisms. We have a forgetful functor $uS^f M_p Z \rightarrow uM_p Z$ which takes $A = \{a_{i,j}\}$ to the sequence

$$a_{0,p} \longrightarrow \cdots \longrightarrow a_{p-1,p}.$$

This functor is an equivalence of categories, with the inverse functor $uM_p Z \rightarrow uS^f M_p Z$ filling out the $\text{Ar}[p]$ diagram from the sequence with the kernels of the maps. These functors then assemble into a simplicial functor $uM_\bullet Z \rightarrow uS^f M_\bullet Z$.

Now π_{n+1} defines a simplicial functor $uS'_\bullet \mathcal{P}_0^{n+1} \rightarrow uS^f M_\bullet Z$. The following theorem fixes the argument in [5] by replacing Theorem 3.10.

Theorem 12.15. *The simplicial functors $uS'_\bullet \mathcal{P}_0^{n+1} \rightarrow uS^f M_\bullet Z \leftarrow uM_\bullet Z$ induce weak equivalences on nerves.*

Proof. Fix p . Since $uM_p Z \rightarrow uS^f M_p Z$ is an equivalence of categories, it induces a weak equivalence on nerves. The proof of Theorem 3.10 in [5, §4] correctly proves that the functor $uS_p \mathcal{P}_0^{n+1} \rightarrow uM_p Z$ induces a weak equivalence on nerves, and the composite functor

$$uS_p \mathcal{P}_0^{n+1} \longrightarrow uM_p Z \longrightarrow uS^f M_p Z$$

is naturally isomorphic to the functor $uS_p \mathcal{P}_0^{n+1} \rightarrow uS^f M_p Z$ in the statement, so that functor also induces a weak equivalence on nerves. \square

We regard the categories $u_q S^f M_p$ as simplicially enriched with discrete mapping spaces and we obtain a connective spectral enrichment $(u_q S^f M_p)^\Gamma$ using objectwise direct sum. Since the functor $u_q M_p Z \rightarrow u_q S^f M_p Z$ is an equivalence of categories, we get a DK-equivalence on the connective spectral enrichments.

Proposition 12.16. *The spectral functor $(u_q M_p Z)^\Gamma \rightarrow (u_q S^f M_p Z)^\Gamma$ is a DK-equivalence.*

Finally, we have everything in place to prove Lemma 12.5.

Proof of Lemma 12.5. Propositions 12.8, 12.9, 12.10, 12.13, and 12.16 imply that the bisimplicial map

$$THH(S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma) \longrightarrow THH((u_q S^f M_p Z)^\Gamma)$$

is a weak equivalence for each fixed p, q . Propositions 12.12 and 12.16 then imply that

$$\|THH(S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma)\| \simeq \|THH((u_\bullet S^f M_\bullet Z)^\Gamma)\| \simeq \|THH((u M_\bullet Z)^\Gamma)\|$$

is contractible. \square

Part 4. Generalization to Waldhausen categories with factorization

13. WEAKLY EXACT FUNCTORS

In previous sections, we imposed stringent hypotheses on our categories and functors. In this section and the following sections, we relax these hypotheses and extend the theory. We begin in this section by generalizing the maps we consider.

While (for now) we still consider functors that preserve the simplicial enrichment, we drop the hypothesis that the functor is exact, and substitute the up to weak equivalence version of this hypothesis that the functor is “weakly exact” [6, §2]. For Waldhausen categories that admit functorial factorization of weak cofibrations (FFWC), a weakly exact functor is the minimum structure necessary to induce a map on K -theory. The purpose of this section is to explain the proof of the following theorem, which provides the corresponding result in our setting.

Theorem 13.1. *Let \mathcal{C} and \mathcal{D} be simplicially enriched Waldhausen categories and assume that the underlying Waldhausen category of \mathcal{D} admits FFWC. Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a simplicially enriched functor that restricts to a based weakly exact functor on the underlying Waldhausen categories, then it induces a map*

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\mathcal{D})$$

in the homotopy category of cyclotomic spectra. This map is compatible with the cyclotomic trace in that the following diagram commutes in the stable category.

$$\begin{array}{ccccc} KC & \xrightarrow{\text{trc}} & WTC^\Gamma(\mathcal{C}) & \longrightarrow & WTHH^\Gamma(\mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow \\ K\mathcal{D} & \xrightarrow{\text{trc}} & WTC^\Gamma\mathcal{D} & \longrightarrow & WTHH^\Gamma(\mathcal{D}) \end{array}$$

In the case of enhanced simplicially enriched Waldhausen categories, we have the following version of the previous theorem.

Theorem 13.2. *Let \mathcal{A} and \mathcal{B} be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories \mathcal{C} and \mathcal{D} respectively. If $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is a simplicially enriched functor that sends \mathcal{A} into \mathcal{B} and*

restricts to a based weakly exact functor on the underlying Waldhausen categories, then it induces a map in the homotopy category of cyclotomic spectra

$$WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{B})$$

making the following diagram commute in the homotopy category of cyclotomic spectra.

$$\begin{array}{ccc} WTHH^\Gamma(\mathcal{A}) & \longrightarrow & WTHH(\mathcal{A}) \\ \downarrow & & \downarrow \\ WTHH^\Gamma(\mathcal{B}) & \longrightarrow & WTHH(\mathcal{B}) \end{array}$$

We also have the following theorem for natural weak equivalences between enriched weakly exact functors.

Theorem 13.3. *Let ϕ and ϕ' be as in Theorem 13.1 or Theorem 13.2 above. If there is a natural weak equivalence from ϕ to ϕ' , then the induced maps from $WTHH^\Gamma(\mathcal{C})$ to $WTHH^\Gamma(\mathcal{D})$ agree in the homotopy category of cyclotomic spectra and (for Theorem 13.2) the induced maps from $WTHH(\mathcal{A})$ to $WTHH(\mathcal{B})$ agree in the homotopy category of cyclotomic spectra.*

The proof of these theorems requires the S'^M_\bullet construction from Section 4; a weakly exact functor is precisely a functor that is compatible with that construction. We begin with the definition of weakly exact functor.

Definition 13.4 ([6, 2.1]). Let \mathcal{C}_0 and \mathcal{D}_0 be Waldhausen categories. A functor $\phi: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ is *weakly exact* if the initial map $* \rightarrow \phi(*)$ in \mathcal{D}_0 is a weak equivalence and ϕ preserves weak equivalences, weak cofibrations, and homotopy cocartesian squares. We say that a weakly exact functor ϕ is *based* if the initial map $* \rightarrow \phi(*)$ is the identity.

It follows that a functor that preserves weak equivalences will preserve weak cofibrations and homotopy cocartesian squares if and only if it takes cofibrations to weak cofibrations and takes pushouts along cofibrations to homotopy cocartesian squares.

Let

$$\begin{aligned} W'THH^\Gamma \mathcal{C} &= \Omega |THH(S'^M_\bullet \mathcal{C}^\Gamma)| \\ \widetilde{W'THH}^\Gamma \mathcal{C}(n) &= |THH((wS'^{(n)}_{\bullet, \dots, \bullet})^M \mathcal{C}^\Gamma)|. \end{aligned}$$

If \mathcal{A} is an enhanced simplicially enriched Waldhausen category, let

$$\begin{aligned} W'THH \mathcal{A} &= \Omega |THH(S'^M_\bullet \mathcal{A}^S)| \\ \widetilde{W'THH} \mathcal{A}(n) &= |THH((wS'^{(n)}_{\bullet, \dots, \bullet})^M \mathcal{A}^S)|. \end{aligned}$$

Proposition 4.6 now implies the following theorem.

Theorem 13.5. *Let \mathcal{C} be a simplicially enriched Waldhausen category that admits FFWC. The maps of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow W'THH^\Gamma(\mathcal{C}) \quad \text{and} \quad \widetilde{W'THH}^\Gamma(\mathcal{C}) \longrightarrow W'THH^\Gamma(\mathcal{C})$$

are weak equivalences. If \mathcal{A} is an enhanced simplicially enriched Waldhausen category, then the maps of cyclotomic spectra

$$WTHH(\mathcal{A}) \longrightarrow W'THH(\mathcal{A}) \quad \text{and} \quad \widetilde{W'THH}(\mathcal{A}) \longrightarrow W'THH(\mathcal{A})$$

are weak equivalences.

Functoriality of THH in weakly exact functors requires one more twist. Because an exact functor $\mathcal{C}_0 \rightarrow \mathcal{D}_0$ preserves coproducts, an enriched exact functor induces a functor on spectral enrichments. For a weakly exact functor ϕ , the map

$$\phi(c_1) \vee \cdots \vee \phi(c_n) \longrightarrow \phi(c_1 \vee \cdots \vee c_n)$$

is generally not an isomorphism, though it is required to be a weak equivalence. To fix this problem, we use a zigzag with the following construction.

Construction 13.6. For enriched categories \mathcal{C} and \mathcal{D} , and an enriched functor $\phi: \mathcal{C} \rightarrow \mathcal{D}$, let $\phi^*\mathcal{D}$ be the simplicially enriched category whose objects are the objects of \mathcal{C} and whose maps are $\phi^*\mathcal{D}(a, b) = \mathcal{D}(\phi(a), \phi(b))$.

When $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is a simplicially enriched functor that restricts to a based weakly exact functor $\mathcal{C}_0 \rightarrow \mathcal{D}_0$, the category $\phi^*\mathcal{D}$ has two obvious connective spectral enrichments. We write $\phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma}$ for the spectral enrichment prolonged from the Γ -space

$$(\phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma})_q(a, b) = \phi^*\mathcal{D}(a, \bigvee_q b) = \mathcal{D}(\phi(a), \phi(\bigvee_q b))$$

(the wedge denoting the coproduct in \mathcal{C}); the hypothesis that the functor is based ensures that $(\phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma})_0(a, b) = *$. We write $\phi^*\mathcal{D}^{\Gamma}$ for the spectral enrichment prolonged from the Γ -space

$$(\phi^*\mathcal{D}^{\Gamma})_q(a, b) = \mathcal{D}(\phi(a), \bigvee_q \phi(b))$$

(the wedge denoting the coproduct in \mathcal{D}). From the universal property of the coproduct in \mathcal{D} , we see that the weak equivalences

$$(\phi^*\mathcal{D}^{\Gamma})_q(a, b) \longrightarrow (\phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma})_q(a, b)$$

assemble into a weak equivalence of spectra

$$\phi^*\mathcal{D}^{\Gamma}(a, b) \longrightarrow \phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma}(a, b)$$

and induce a DK-equivalence of spectrally enriched categories $\phi^*\mathcal{D}^{\Gamma} \rightarrow \phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma}$. We summarize this as follows.

Theorem 13.7. *Let \mathcal{C} and \mathcal{D} be simplicially enriched Waldhausen categories and let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that is simplicially enriched and based weakly exact. Then we have a zigzag of spectrally enriched functors, with the leftward arrow a DK-equivalence.*

$$\mathcal{C}^{\Gamma} \longrightarrow \phi^*\mathcal{D}_{\mathcal{C}}^{\Gamma} \xleftarrow{\sim} \phi^*\mathcal{D}^{\Gamma} \longrightarrow \mathcal{D}^{\Gamma}$$

The constructions above, using coproducts in \mathcal{C} or in \mathcal{D} , extend to produce spectrally enriched categories and DK-equivalences

$$\phi^*(w_p S'_{q_1, \dots, q_n}(n))^M \mathcal{D}^{\Gamma} \longrightarrow \phi^*(w_p S'_{q_1, \dots, q_n}(n))^M \mathcal{D}_{\mathcal{C}}^{\Gamma},$$

which we assemble into zigzags of multisimplicial spectrally enriched functors

$$w_{\bullet} S'_{\bullet, \dots, \bullet}(n) \mathcal{C}^{\Gamma} \longrightarrow \phi^*(w_{\bullet} S'_{\bullet, \dots, \bullet}(n))^M \mathcal{D}_{\mathcal{C}}^{\Gamma} \xleftarrow{\sim} \phi^*(w_{\bullet} S'_{\bullet, \dots, \bullet}(n))^M \mathcal{D}^{\Gamma} \longrightarrow (w_{\bullet} S'_{\bullet, \dots, \bullet}(n))^M \mathcal{D}^{\Gamma}.$$

The fact that ϕ is based implies that these maps are compatible with the suspension maps

$$\Sigma(w_{\bullet} S'_n(n))^M \mathcal{X}^{\Gamma} \longrightarrow (w_{\bullet} S'_{n+1}(n+1))^M \mathcal{X}^{\Gamma}$$

for $\mathcal{X}^\Gamma = \mathcal{C}^\Gamma, \phi^* \mathcal{D}_\mathcal{C}^\Gamma, \phi^* \mathcal{D}^\Gamma, \mathcal{D}^\Gamma$. Writing

$$\begin{aligned} W'THH^\Gamma(\phi^* \mathcal{D}) &= \Omega |THH(\phi^* S'_\bullet{}^M \mathcal{D}^\Gamma)| \\ \widetilde{W'THH}^\Gamma(\phi^* \mathcal{D})(n) &= |THH(\phi^*(wS'_{\bullet, \dots, \bullet}{}^{(n)})^M \mathcal{D}^\Gamma)| \end{aligned}$$

and

$$\begin{aligned} W'THH_\mathcal{C}^\Gamma(\phi^* \mathcal{D}) &= \Omega |THH(\phi^* S'_\bullet{}^M \mathcal{D}_\mathcal{C}^\Gamma)| \\ \widetilde{W'THH}_\mathcal{C}^\Gamma(\phi^* \mathcal{D})(n) &= |THH(\phi^*(wS'_{\bullet, \dots, \bullet}{}^{(n)})^M \mathcal{D}_\mathcal{C}^\Gamma)|, \end{aligned}$$

this proves the following theorem.

Theorem 13.8. *Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a simplicially enriched functor that restricts to a based weakly exact functor $\mathcal{C}_0 \rightarrow \mathcal{D}_0$. Then we have the following maps of cyclotomic spectra*

$$\begin{array}{ccc} WTHH^\Gamma(\mathcal{C}) & & \widetilde{WTHH}^\Gamma(\mathcal{C}) \\ \downarrow & & \downarrow \\ W'THH_\mathcal{C}^\Gamma(\phi^* \mathcal{D}) & & \widetilde{W'THH}_\mathcal{C}^\Gamma(\phi^* \mathcal{D}) \\ \sim \uparrow & & \sim \uparrow \\ W'THH^\Gamma(\phi^* \mathcal{D}) & & \widetilde{W'THH}^\Gamma(\phi^* \mathcal{D}) \\ \downarrow & & \downarrow \\ W'THH^\Gamma(\mathcal{D}) & & \widetilde{W'THH}^\Gamma(\mathcal{D}) \\ \uparrow & & \uparrow \\ WTHH^\Gamma(\mathcal{D}) & & \widetilde{WTHH}^\Gamma(\mathcal{D}) \end{array}$$

and the upward maps marked “ \sim ” are weak equivalences. If \mathcal{D}_0 admits FFWC, then all upward maps are weak equivalences.

For Theorem 13.1, we have the cyclotomic trace induced by the inclusion of objects, producing the commutative diagram

$$\begin{array}{ccccc} K(\mathcal{C}) & \longrightarrow & \widetilde{WTC}^\Gamma(\mathcal{C}) & \longrightarrow & \widetilde{WTHH}^\Gamma(\mathcal{C}) \\ & \searrow & \downarrow & & \downarrow \\ & & \widetilde{W'TC}_\mathcal{C}^\Gamma(\phi^* \mathcal{D}) & \longrightarrow & \widetilde{W'THH}_\mathcal{C}^\Gamma(\phi^* \mathcal{D}) \\ & & \sim \uparrow & & \sim \uparrow \\ & & \widetilde{W'TC}^\Gamma(\phi^* \mathcal{D}) & \longrightarrow & \widetilde{W'THH}^\Gamma(\phi^* \mathcal{D}) \\ & & \downarrow & & \downarrow \\ K'(\mathcal{D}) & \longrightarrow & \widetilde{W'THH}^\Gamma(\mathcal{D}) & \longrightarrow & \widetilde{W'THH}^\Gamma(\mathcal{D}) \\ \sim \uparrow & & \sim \uparrow & & \sim \uparrow \\ K(\mathcal{D}) & \longrightarrow & \widetilde{WTC}^\Gamma(\mathcal{D}) & \longrightarrow & \widetilde{WTHH}^\Gamma(\mathcal{D}) \end{array}$$

where here $K'(\mathcal{D})$ denotes K -theory constructed from the S'_\bullet construction. This completes the proof the Theorem 13.1.

For Theorem 13.2, we construct non-connective enrichments on $\phi^* \mathcal{B}$ by

$$\begin{aligned} \phi^* \mathcal{B}_\mathcal{C}^S(a, b)(n) &= \mathcal{D}(\phi(a), \phi(\Sigma^n b)) \\ \phi^* \mathcal{B}^S(a, b)(n) &= \mathcal{D}(\phi(a), \Sigma^n \phi(b)). \end{aligned}$$

The hypothesis that $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is enriched induces the map $\Sigma^n \phi(b) \rightarrow \phi(\Sigma^n b)$, and the hypothesis that ϕ preserves homotopy cocartesian squares implies that the map is a weak equivalence. It follows that the spectrally enriched functor $\phi^* \mathcal{B}^S \rightarrow \phi^* \mathcal{B}_{\mathcal{C}}^S$ is a DK-equivalence. This gives us the non-connective analogue of Theorem 13.7.

Theorem 13.9. *Let \mathcal{A} and \mathcal{B} be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories \mathcal{C} and \mathcal{D} , respectively, and let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that is simplicially enriched, based weakly exact, and restricts to a functor $\mathcal{A} \rightarrow \mathcal{B}$. Then we have a zigzag of spectrally enriched functors, with the leftward arrow a DK-equivalence.*

$$\mathcal{A}^S \rightarrow \phi^* \mathcal{B}_{\mathcal{C}}^S \xleftarrow{\sim} \phi^* \mathcal{B}^S \rightarrow \mathcal{B}^S$$

Likewise, writing

$$\begin{aligned} W'THH(\phi^* \mathcal{B}) &= \Omega |THH(\phi^* S'_{\bullet}{}^M \mathcal{B}^S)| \\ \widetilde{W'THH}(\phi^* \mathcal{B})(n) &= |THH(\phi^*(wS'_{\bullet, \dots, \bullet}{}^{(n)} \mathcal{B}^S))| \end{aligned}$$

and

$$\begin{aligned} W'THH_{\mathcal{A}}(\phi^* \mathcal{B}) &= \Omega |THH(\phi^* S'_{\bullet}{}^M \mathcal{B}_{\mathcal{A}}^S)| \\ \widetilde{W'THH}_{\mathcal{A}}(\phi^* \mathcal{B})(n) &= |THH(\phi^*(wS'_{\bullet, \dots, \bullet}{}^{(n)} \mathcal{B}_{\mathcal{A}}^S))|, \end{aligned}$$

we obtain non-connective analogue of Theorem 13.8.

Theorem 13.10. *Let \mathcal{A} and \mathcal{B} be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories \mathcal{C} and \mathcal{D} , respectively, and let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that is simplicially enriched, based weakly exact, and restricts to a functor $\mathcal{A} \rightarrow \mathcal{B}$. Then we have the following maps of cyclotomic spectra*

$$\begin{array}{ccc} WTHH(\mathcal{C}) & & \widetilde{WTHH}(\mathcal{C}) \\ \downarrow & & \downarrow \\ W'THH_{\mathcal{C}}(\phi^* \mathcal{D}) & & \widetilde{W'THH}_{\mathcal{C}}(\phi^* \mathcal{D}) \\ \sim \uparrow & & \sim \uparrow \\ W'THH(\phi^* \mathcal{D}) & & \widetilde{W'THH}(\phi^* \mathcal{D}) \\ \downarrow & & \downarrow \\ W'THH(\mathcal{D}) & & \widetilde{W'THH}(\mathcal{D}) \\ \uparrow & & \uparrow \\ WTHH(\mathcal{D}) & & \widetilde{WTHH}(\mathcal{D}) \end{array}$$

and the upward maps marked “ \sim ” are weak equivalences. If \mathcal{D}_0 admits FFWC, then all upward maps are weak equivalences.

Finally for Theorem 13.3, choosing a natural weak equivalence from ϕ to ϕ' , we obtain a simplicially enriched and weakly exact functor Φ from \mathcal{C} to $w_1 \mathcal{D}$. We can construct simplicially enriched categories

$$\Phi^*(S'_n w_1)^M \mathcal{D}$$

and we obtain the zigzag

$$\begin{aligned} THH(S_\bullet \mathcal{C}^\Gamma) &\longrightarrow THH(\Phi^*(S'_\bullet w_1)^M \mathcal{D}_\mathcal{C}^\Gamma) \xleftarrow{\sim} THH(\Phi^*(S'_\bullet w_1)^M \mathcal{D}^\Gamma) \\ &\longrightarrow THH((S'_\bullet w_1)^M \mathcal{D}^\Gamma) \xleftarrow{\sim} THH(S'_\bullet \mathcal{D}^\Gamma), \end{aligned}$$

and a similar zigzag in the non-connective case (when it applies).

14. EMBEDDING IN SIMPLICIALLY TENSORED WALDHAUSEN CATEGORIES

In previous sections we worked under the stringent compatibility hypotheses in our definition of a simplicially enriched Waldhausen category. In this section, we show how to produce a DK-compatible simplicially enriched Waldhausen category from a Waldhausen category satisfying a certain technical hypothesis. This construction is natural in the homotopy category of cyclotomic spectra for weakly exact functors; we prove the following theorem.

Theorem 14.1. *Let \mathcal{C} be a Waldhausen category that admits a homotopy calculus of left fractions [9]. Then there exists a DK-compatible simplicially enriched Waldhausen category $\tilde{\mathcal{C}}$ and a based weakly exact functor $\tilde{\iota}: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ that is a DK-equivalence (on simplicial localizations). Moreover:*

- (i) $WTHH^\Gamma(\tilde{\mathcal{C}})$ is a functor from the category of Waldhausen categories and weakly exact maps to the homotopy category of cyclotomic spectra.
- (ii) As a map in the stable category, $K(\mathcal{C}) \rightarrow K(\tilde{\mathcal{C}})$ is natural in exact functors of \mathcal{C} .
- (iii) As a map in the stable category, the cyclotomic trace $K(\tilde{\mathcal{C}}) \rightarrow WTHH^\Gamma(\tilde{\mathcal{C}})$ is natural in weakly exact functors of \mathcal{C} .
- (iv) $\tilde{\mathcal{C}}$ admits FFWC.
- (v) If \mathcal{C} is a DK-compatible simplicially enriched Waldhausen category then $\tilde{\iota}$ is naturally weakly equivalent to a simplicially enriched functor $\tilde{\iota}'$, which is also based weakly exact.
- (vi) If \mathcal{C} can be given the structure of an enhanced simplicially enriched Waldhausen category, then $\tilde{\iota}'$ induces DK-equivalence $S_n \mathcal{C} \rightarrow S_n^M \tilde{\mathcal{C}}$ for all n and so induces a weak equivalence on $WTHH^\Gamma$.

In the context of part (v), Theorem 13.7 gives a zigzag of spectrally enriched functors relating \mathcal{C}^Γ and $\tilde{\mathcal{C}}^\Gamma$, all of which are DK-equivalences in this case.

As we showed in [6, §5, App. A], a Waldhausen category that admits factorization (every map factors as a cofibration followed by a weak equivalence) and any closed Waldhausen subcategory of such a category in particular admits a homotopy calculus of left fractions. In this context, we can also produce an enhanced exact Waldhausen category.

Theorem 14.2. *Let \mathcal{C} be a Waldhausen category that admits factorization and let \mathcal{A} be a closed Waldhausen subcategory. Let $\check{\mathcal{A}}$ be the full subcategory of $\tilde{\mathcal{C}}$ of objects weakly equivalent to objects from \mathcal{A} . Then $\tilde{\mathcal{C}}$ is a simplicially tensored Waldhausen category, $\check{\mathcal{A}}$ is a closed Waldhausen subcategory, and the induced based weakly exact functor $\tilde{\iota}: \mathcal{A} \rightarrow \check{\mathcal{A}}$ is a DK-equivalence. Moreover:*

- (i) $WTHH(\check{\mathcal{A}})$ is a functor from the category of pairs (Waldhausen category, closed Waldhausen subcategory) and weakly exact maps to the homotopy category of cyclotomic spectra.

- (ii) *There is a based weakly exact functor $\check{\jmath}: \check{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ such that $\check{\jmath} \circ \check{\imath}$ is naturally weakly equivalent to $\tilde{\imath}$. (In particular, $\check{\jmath}$ is a DK-equivalence.)*
- (iii) *The map of cyclotomic spectra $WTHH^\Gamma(\check{\mathcal{A}}) \rightarrow WTHH^\Gamma(\tilde{\mathcal{A}})$ induced by $\check{\jmath}$ is a weak equivalence and natural in the homotopy category of cyclotomic spectra.*

In the context of the previous theorem, when \mathcal{C} is a simplicially tensored Waldhausen category, \mathcal{A} is an enhanced simplicially enriched Waldhausen category, and part (v) of Theorem 14.1 gives us a based weakly exact simplicially enriched functor $\check{\imath}': \mathcal{A} \rightarrow \check{\mathcal{A}}$, weakly equivalent to $\check{\imath}$; namely, $\check{\imath}'$ is the restriction to \mathcal{A} of $\check{\imath}': \mathcal{C} \rightarrow \tilde{\mathcal{C}}$. Theorem 13.9 then produces a zigzag of spectrally enriched functors between \mathcal{A}^S and $\check{\mathcal{A}}^S$, all of which are DK-equivalences in this case.

The proof of the previous theorems works by embedding \mathcal{C} in a simplicial model category in which all objects are fibrant. We do this using a variant of a presheaf construction in Toën and Vezzosi [27] to define the K -theory of a simplicial category. Because of the nature of the construction, the naturality with respect to functors that preserve weak equivalences is somewhat complicated (as indicated by what is claimed and perhaps more by what is not claimed in the previous theorems). In the following discussion, let LC denote the simplicial category obtained as the Dwyer-Kan hammock simplicial localization of \mathcal{C} with respect to the weak equivalences in the given Waldhausen structure.

Definition 14.3. Let $SF(LC)$ denote the category of simplicial functors from LC to based simplicial sets taking values in a fixed but sufficiently large cardinal depending on \mathcal{C} . We regard $SF(LC)$ as a simplicial model category using the injective model structure [12], where cofibrations and weak equivalences are defined object-wise and fibrations are defined by the right-lifting property with respect to the acyclic cofibrations; in this model structure, all objects are cofibrant. The opposite category $(SF(LC))^{\text{op}}$ then has the opposite simplicial model structure and all objects are fibrant.

Since the cofibrations in $SF(LC)$ are the injections, it is clear that $SF(LC)$ satisfies the pushout-product axiom, which is one of the equivalent forms of Quillen's SM7; in other words, $SF(LC)$ is a simplicial model category. It follows that $(SF(LC))^{\text{op}}$ is likewise a simplicial model category. Heller [12, §4] shows that the injective model structure has functorial factorizations, and in particular, we have a fibrant replacement functor in $SF(LC)$. In $(SF(LC))^{\text{op}}$, this gives functorial factorization and a cofibrant approximation functor. It will be useful for us to have these as simplicial functors and to preserve the zero object $*$. We prove the following lemma at the end of the section.

Lemma 14.4. *The category $SF(LC)$ admits simplicial endo-functors P^c and I^f such that P^c is a cofibrant approximation functor for the projective model structure, I^f is a fibrant approximation functor for the injective model structure, and $P^c(*) = * = I^f(*)$.*

The full subcategory of cofibrant objects in $(SF(LC))^{\text{op}}$ inherits the structure of a Waldhausen category.

Definition 14.5. Let $\tilde{\mathcal{C}}$ be the full subcategory of $(SF(LC))^{\text{op}}$ consisting of cofibrant objects weakly equivalent to the opposite of a corepresentable in the image of \mathcal{C} , i.e., weakly equivalent to a functor of the form $LC(x, -)$, where x is an object of

\mathcal{C} . When \mathcal{A} is a closed Waldhausen subcategory of \mathcal{C} , let $\check{\mathcal{A}}$ be the full subcategory of $(SF(LC))^{\text{op}}$ consisting of cofibrant objects weakly equivalent to the opposite of a corepresentable of an object \mathcal{A} .

As observed in Example 1.5, $\tilde{\mathcal{C}}$ becomes a DK-compatible simplicially enriched Waldhausen category when given the Waldhausen structure induced by the model structure. The Yoneda embedding

$$Y_{\mathcal{C}}: x \mapsto LC(x, -)$$

gives us a functor $Y_{\mathcal{C}}$ from \mathcal{C} to $(SF(LC))^{\text{op}}$ that we can compose with I^f to obtain a functor $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$. We showed in [6, 6.2] that under the hypothesis of homotopy calculus of left fractions, the simplicial localization mapping spaces take homotopy cocartesian squares to homotopy cartesian squares, and hence to homotopy cocartesian squares in $(SF(LC))^{\text{op}}$. It follows that $I^f Y_{\mathcal{C}}$ is a weakly exact functor $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$ and a DK-equivalence. It is not, however, a based weakly exact functor as the zero object of \mathcal{C} is generally not a zero object in LC . On the other hand, $LC(*, -) \rightarrow LC(x, -)$ is an objectwise injection (as it is split by the map $LC(x, -) \rightarrow LC(*, -)$), and so the based functor

$$Y'_{\mathcal{C}}: x \mapsto LC(x, -)/LC(*, -)$$

is weakly equivalent to $Y_{\mathcal{C}}$ and hence is a based weakly exact functor and DK-equivalence. This proves the following proposition.

Proposition 14.6. *Let \mathcal{C} be a Waldhausen category that admits a homotopy calculus of left fractions. Then the functor $\tilde{\iota} = I^f Y'_{\mathcal{C}}: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is a based weakly exact functor and a DK-equivalence.*

When \mathcal{C} is a simplicially enriched Waldhausen category, $\tilde{\iota}$ is a simplicial functor $LC \rightarrow \tilde{\mathcal{C}}$ but generally not a simplicial functor $\mathcal{C}_{\bullet} \rightarrow \tilde{\mathcal{C}}$. We can regard the functor

$$x \mapsto \text{diag } LC_{\bullet}(x, -)/LC_{\bullet}(*, -)$$

as a simplicial functor from \mathcal{C} to $(SF(LC))^{\text{op}}$. Composing with I^f , we get a simplicial functor $\tilde{\iota}': \mathcal{C} \rightarrow \tilde{\mathcal{C}}$. The inclusion of LC_0 in LC_{\bullet} induces a natural transformation $\tilde{\iota} \rightarrow \tilde{\iota}'$, which is a natural weak equivalence when \mathcal{C} is DK-compatible (by definition). This proves the following proposition.

Proposition 14.7. *If \mathcal{C} is a DK-compatible simplicially enriched Waldhausen category, then $\tilde{\iota}$ is weakly equivalent to a simplicial functor, which is also a based weakly exact DK-equivalence.*

When \mathcal{C} is a DK-compatible simplicially enriched Waldhausen category, just as in Proposition 3.4, looking at the formula for mapping spectra in $S_n \mathcal{C}$ and $S'_n{}^M \tilde{\mathcal{C}}$, we see that $\tilde{\iota}'$ induces a DK-embedding $S_n \mathcal{C} \rightarrow S'_n{}^M \tilde{\mathcal{C}}$. If we assume the hypothesis of part (vi), then \mathcal{C} admits tensors with $\Delta[1]$, and for weak cofibration $x \rightarrow y$, the map $x \vee y \rightarrow (x \otimes \Delta[1]) \cup_{x \otimes \{1\}} y$ is a cofibration, i.e., \mathcal{C} has functorial mapping cylinders for weak cofibrations in the terminology of [6, 2.6]. Since in any simplicially enriched Waldhausen category, weak equivalences are closed under retracts, we can apply [6, 6.1] to characterize the weak cofibrations in \mathcal{C} as precisely those maps whose images in $\tilde{\mathcal{C}}$ are weak cofibrations. Moreover, tensors with generalized intervals exist in \mathcal{C} , and arguing as in the proof of Proposition 3.6, we see that every object of $S'_n{}^M \tilde{\mathcal{C}}$ is

weakly equivalent to the image of an object of $S'_n\mathcal{C}$, i.e., that the DK-embedding is a DK-equivalence. The induced map (from Theorem 13.1)

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\tilde{\mathcal{C}})$$

is then a weak equivalence.

Now drop the assumption that \mathcal{C} is simplicially enriched, and assume instead that \mathcal{C} admits factorization. Then Waldhausen [28, p. 357] shows that we can form homotopy colimits in \mathcal{C} over diagrams in finite partially ordered sets as iterated pushouts over cofibrations. Since any finite simplicial set is weakly equivalent to the nerve of a finite partially ordered set, it follows that for any weakly corepresentable C and any finite simplicial set X , the simplicial functor C^X is also weakly corepresentable. This proves the following proposition.

Proposition 14.8. *If \mathcal{C} admits factorization then $\tilde{\mathcal{C}}$ is a simplicially tensored Waldhausen category.*

We also have the corresponding proposition for closed Waldhausen subcategories.

Proposition 14.9. *If \mathcal{A} is a closed Waldhausen subcategory of \mathcal{C} , then $\check{\mathcal{A}} \subset \tilde{\mathcal{C}}$ is an enhanced simplicial Waldhausen category and $\check{i}: \mathcal{A} \rightarrow \check{\mathcal{A}}$ is a based weakly exact functor and a DK-equivalence on simplicial localizations.*

We obtain the functor $\check{j}: \check{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ as the restriction to $\check{\mathcal{A}}$ of the functor $I_{\mathcal{A}}^f \circ R_{\mathcal{C}}^{\mathcal{A}}$, where $R_{\mathcal{C}}^{\mathcal{A}}$ denotes the functor $(SF(LC))^{\text{op}} \rightarrow (SF(L\mathcal{A}))^{\text{op}}$ obtained by restricting an LC diagram to $L\mathcal{A}$ and $I_{\mathcal{A}}^f$ denotes the endo-functor I^f in $\tilde{\mathcal{A}}$. Writing $Y'_{\mathcal{A}}$ and Y'_C for the modified Yoneda embeddings as above, then

$$\check{j} \circ \check{i} = I_{\mathcal{A}}^f R_{\mathcal{C}}^{\mathcal{A}} I_{\mathcal{C}}^f Y'_C \quad \text{and} \quad \tilde{i} = I_{\mathcal{A}}^f Y'_{\mathcal{A}}.$$

Under the hypothesis of homotopy calculus of left fractions, the natural map $Y'_{\mathcal{A}} \rightarrow R_{\mathcal{C}}^{\mathcal{A}} Y'_C$ in $SF(L\mathcal{A})$ is a weak equivalence; combining this with the canonical weak equivalence $\text{Id} \rightarrow I_{\mathcal{C}}^f$ in $SF(LC)$ and reversing arrows to work in $(SF(L\mathcal{A}))^{\text{op}}$ gives natural weak equivalences

$$\check{j} \circ \check{i} = I_{\mathcal{A}}^f R_{\mathcal{C}}^{\mathcal{A}} I_{\mathcal{C}}^f Y'_C \longrightarrow I_{\mathcal{A}}^f R_{\mathcal{C}}^{\mathcal{A}} Y'_C \longrightarrow I_{\mathcal{A}}^f Y'_{\mathcal{A}} = \tilde{i}$$

in $\tilde{\mathcal{A}}$.

The previous observations, propositions, and definitions cover all of the statements in Theorems 14.1 and 14.2 except for the naturality statements. The next result begins the study of naturality.

Theorem 14.10. *Let \mathcal{C} and \mathcal{C}' be Waldhausen categories that admit homotopy calculi of left fractions, and let $\phi: \mathcal{C} \rightarrow \mathcal{C}'$ be a weakly exact functor. Then there exists a simplicial functor $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}'}$ that restricts to a based weakly exact functor of the underlying Waldhausen categories and makes the diagram of functors*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \tilde{\mathcal{C}} \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \mathcal{C}' & \longrightarrow & \tilde{\mathcal{C}'} \end{array}$$

commute up to a zigzag of natural weak equivalences.

If \mathcal{A} and \mathcal{A}' are closed Waldhausen subcategories of \mathcal{C} and \mathcal{C}' (respectively) and ϕ restricts to a functor from \mathcal{A} to \mathcal{A}' , then the functor $\tilde{\phi}$ restricts to a functor $\tilde{\phi}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}'}$ making the diagram of functors

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \tilde{\mathcal{A}} \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \mathcal{A}' & \longrightarrow & \tilde{\mathcal{A}'} \end{array}$$

commute up to a zigzag of natural weak equivalences.

We prove this theorem below, but first state the following corollary.

Corollary 14.11. *Let \mathcal{C} and \mathcal{C}' be Waldhausen categories that admit homotopy calculi of left fractions, and let $\phi: \mathcal{C} \rightarrow \mathcal{C}'$ be a weakly exact functor. If ϕ induces a DK-equivalence on passage to simplicial localizations, then the functor $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}'}$ is a DK-equivalence. Moreover, $\tilde{\phi}$ and (when appropriate) $\check{\phi}$ induce an equivalence of cyclotomic spectra on $WTHH^\Gamma$ and $WTHH$, respectively.*

The proof of Theorem 14.10 combines the simplicially enriched cofibrant and fibrant approximation functors with left Kan extension. Fix the functor $\phi: \mathcal{C} \rightarrow \mathcal{C}'$. Left Kan extension gives rise to a functor $\text{Lan}_\phi: SF(LC) \rightarrow SF(LC')$ and we let $\tilde{\phi}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}'}$ be the composite functor

$$SF(LC) \xrightarrow{P^c} SF(LC) \xrightarrow{\text{Lan}_\phi} SF(LC') \xrightarrow{I^f} SF(LC').$$

By construction $\tilde{\phi}$ preserves weak equivalences and is equipped with a zig-zag of natural weak equivalences connecting $\tilde{i} \circ \tilde{\phi}$ to $\phi \circ \tilde{i}$. This completes the proof of Theorem 14.10.

Most of Corollary 14.11 follows immediately from Theorem 14.10. To see that $\tilde{\phi}$ induces a weak equivalence on $WTHH^\Gamma$, we need to see that the induced functor $S_n \tilde{\mathcal{C}} \rightarrow S_n^M \tilde{\mathcal{C}'}$ is a DK-equivalence. The argument for Proposition 3.6 adapts to the current context to complete the proof.

The proof of the naturality statements in Theorems 14.1 and 14.2 now follow from an easy check that functors $\tilde{\phi}$ compose as expected up to a zigzag of natural weak equivalences. Somewhat more work shows that this construction actually preserves composition up to coherent homotopy; we defer this to a future paper.

Finally, we need to prove Lemma 14.4. The specifics of the simplicial category LC play no role: the lemma holds for the category of simplicial functors from any small simplicial category \mathcal{D} to based simplicial sets, and we argue in this context. We prove the following lemma, of which Lemma 14.4 is a special case.

Lemma 14.12. *Let \mathcal{D} be a small simplicial category and let $\mathfrak{S}_*^\mathcal{D}$ denote the category of simplicial functors from \mathcal{D} to based simplicial sets. Then the projective and injective model structures both admit factorization functors that are simplicial functors and that send the identity on $*$ to the factorization $* = * = *$.*

The most basic case is when \mathcal{D} is the trivial category and $\mathfrak{S}_*^\mathcal{D}$ is the category of based simplicial sets. Let C denote the set of generating cofibrations ($\partial\Delta[n] \rightarrow \Delta[n]$) and let A denote the set of generating acyclic cofibrations ($\Lambda_j[n] \rightarrow \Delta[n]$). Then the usual construction of the factorization functors uses the small objects argument as follows. Given $f: x \rightarrow y$, the factorization of f as an acyclic cofibration

$x \rightarrow x'$ followed by a fibration $x' \rightarrow y$ is constructed as $x' = \operatorname{colim} x'_n$, where $x'_0 = x$ and inductively x'_{n+1} is constructed as the pushout

$$x'_{n+1} = x'_n \cup_{\coprod a} (\coprod b)$$

where the coproduct is over commutative diagrams

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ x'_n & \longrightarrow & y \end{array}$$

with $i: a \rightarrow b$ ranging over the elements of A . The version we need for Lemma 14.12 instead uses the based simplicial set of maps in place of the set of maps above: We construct x'_n inductively as the pushout

$$x'_{n+1} = x'_n \cup_{\coprod a \wedge D_i} (\coprod b \wedge D_i)$$

where the coproduct is over the elements $i: a \rightarrow b$ in A and

$$D_i = \mathfrak{S}_*(a, x'_n) \times_{\mathfrak{S}_*(a, y)} \mathfrak{S}_*(b, y)$$

is the based simplicial set of commutative diagrams of the form

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ x'_n & \longrightarrow & y. \end{array}$$

The induced map $x'_n \rightarrow x'_{n+1}$ and the colimit map $x \rightarrow x'$ is an injection and weak equivalence and the map $x' \rightarrow y$ is a fibration. Moreover, this functor is clearly a simplicial functor into the appropriate diagram category. The analogous construction using C instead of A constructs the other factorization. When applied to the identity map on the trivial based simplicial set $*$, each D_i is the trivial based simplicial set $*$, and so we get that each map $* \cong x_n \rightarrow x'_{n+1}$ and $x_{n+1} \rightarrow y = *$ is an isomorphism. Thus, (replacing the factorization functors with naturally isomorphic functors if necessary), we have that the factorization of $* = *$ is $* = * = *$.

A slight modification of the factorization functors in Heller [12] construct the factorizations in the general case. Let $\mathfrak{S}_*^{\operatorname{Ob} \mathcal{D}}$ denote the simplicial category $\prod_{\operatorname{Ob} \mathcal{D}} \mathfrak{S}_*$, and (following the notation in [12]), let J^* denote the forgetful functor from $\mathfrak{S}_*^{\mathcal{D}}$ to $\mathfrak{S}_*^{\operatorname{Ob} \mathcal{D}}$ that remembers just the objects in the diagram (values of the functor) and forgets the maps. Let J_P be its left adjoint; since we are working in based simplicial sets, $J_P X$ is the simplicial functor

$$J_P X = \bigvee_{c \in \operatorname{Ob} \mathcal{D}} X(c) \wedge \mathcal{D}(c, -)_+.$$

Likewise, let J^I be the right adjoint of J^* ,

$$J^I X = \prod_{c \in \operatorname{Ob} \mathcal{D}} X(c)^{\mathcal{D}(-, c)},$$

where $X(c)^{\mathcal{D}(-, c)}$ denotes the based simplicial set of unbased simplicial maps from $\mathcal{D}(-, c)$ to $X(c)$. We note that for any X , $J_P X$ is cofibrant in the projective model structure and more generally, J_P sends (objectwise) cofibrations and acyclic cofibrations in $\mathfrak{S}_*^{\operatorname{Ob} \mathcal{D}}$ to cofibrations and acyclic cofibrations in the projective model structure on $\mathfrak{S}_*^{\mathcal{D}}$. Likewise J^I sends (objectwise) fibrations and acyclic fibrations to fibrations and acyclic fibrations in the injective model structure on $\mathfrak{S}_*^{\mathcal{D}}$.

The factorization functors for the projective model structure are constructed as follows. For $f: X \rightarrow Y$, let $Z_0 = X$ and construct Z_{n+1} inductively as follows. First factor $J^*Z_n \rightarrow J^*Y$ objectwise

$$J^*Z_n \longrightarrow W_n \longrightarrow J^*Y$$

using the simplicial factorization functor (for the appropriate factorization) on based simplicial sets constructed above, and let Z_{n+1} be the pushout

$$Z_{n+1} = Z_n \cup_{J_P J^*Z_n} J_P W_n,$$

with the factorization $Z_{n+1} \rightarrow Y$ induced by the map $J_P W_n \rightarrow Y$. Letting $Z = \text{colim } Z_n$, we get a factorization $X \rightarrow Z \rightarrow Y$, with the map $X \rightarrow Z$ a cofibration or acyclic cofibration (as appropriate) in the projective model structure. We note that the underlying map in $\mathfrak{S}_*^{\text{Ob } \mathcal{D}}$ from J^*Z_n to J^*Z_{n+1} factors through W_n . It follows that we can identify J^*Z as $\text{colim } W_n$ and the underlying map $J^*Z \rightarrow J^*Y$ in $\mathfrak{S}_*^{\text{Ob } \mathcal{D}}$ as the colimit of the maps $W_n \rightarrow J^*Y$. Since by construction, these maps are objectwise acyclic fibrations or fibrations of simplicial sets, the map $J^*Z \rightarrow J^*Y$ is an objectwise acyclic fibration or fibration as required. We note that when $X = * = Y$, by construction each W_n is $*$ and $J_P W_n$ is isomorphic to $*$, and so we end up with both factorizations of $* = *$ as $* = * = *$.

The factorization functors on the injective model structure are precisely dual. We start with $Z_0 = Y$, and inductively construct Z_{n+1} as follows. Using the appropriate objectwise factorization functor, we factor $J^*X \rightarrow J^*Z_n$ in $\mathfrak{S}_*^{\text{Ob } \mathcal{D}}$ as

$$J^*X \longrightarrow W_n \longrightarrow J^*Z_n,$$

and we define Z_{n+1} as the pullback

$$Z_{n+1} = Z_n \times_{J^I J^*Z_n} J^I W_n.$$

We let $Z = \lim Z_n$ and get a factorization $X \rightarrow Z \rightarrow Y$ with $Z \rightarrow Y$ by construction a fibration or acyclic fibration (as appropriate) in the injective model structure. Again looking at the underlying map in $\mathfrak{S}_*^{\text{Ob } \mathcal{D}}$, we see that the map $X \rightarrow Z$ is an objectwise acyclic cofibration or cofibration as appropriate. Again, the factorization of $* = *$ becomes $* = * = *$. This completes the proof of Lemma 14.12.

15. SPECTRAL CATEGORIES AND WALDHAUSEN CATEGORIES

The work of the previous section showed how to associate a spectral category to any well-behaved Waldhausen category. On the other hand, given a spectral category \mathcal{C} , we can produce a simplicially tensored Waldhausen category by passage to the Waldhausen category $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ of “finite cell right \mathcal{C} -modules” described below. In this section we show that when \mathcal{C} is pretriangulated (has triangulated homotopy category), the spectral category associated to $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ in Definition 14.5 recovers the original spectral category \mathcal{C} up to DK-equivalence.

As a general principal, it does not matter which modern category of spectra we use as a model when discussing small spectral categories. The monoidal Quillen equivalences relating the various categories of diagram spectra and EKMM S -modules [15, 16, 21] allow us to convert a spectral category on any of these models to one on any other. In particular, the following theorem is an easy consequence of the work of [23] (extended by the techniques of [11] for dealing with non-cofibrant units that arise there).

Theorem 15.1. *Fix a set of objects O . For \mathcal{S} a modern category of spectra from [16] or [11], let $SO\text{-Cat}$ denote the category of \mathcal{S} -enriched categories with object set O and functors that are the identity on the object set O . Then:*

- (i) *The category $SO\text{-Cat}$ forms a closed model category where the weak equivalences and fibrations are the functors that induce a weak equivalence or positive fibration, respectively, on mapping spectra.*
- (ii) *The monoidal Quillen equivalences from [15, 16, 21] induce Quillen equivalences between the categories $SO\text{-Cat}$ for the various \mathcal{S} .*

Because of this theorem, without loss of generality, we can assume that our spectral category \mathcal{C} comes enriched in EKMM S -modules, which have the technical advantage that every object is fibrant. On the other hand, since our goal is to compare with the non-connective enrichment of a simplicially tensored Waldhausen category, our comparison must be between spectral categories enriched in symmetric spectra. Again, we use the previous theorem. Spectral categories enriched in EKMM S -modules are always fibrant in the model structure of the previous theorem, so the associated spectral category enriched in symmetric spectra has (the same object set and) mapping spectra $\Phi\mathcal{C}(x, y)$, where Φ is the lax symmetric monoidal right adjoint functor from EKMM S -modules to symmetric spectra defined in [21]. Specifically, for an EKMM S -module X ,

$$\Phi X(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, X).$$

Here \mathcal{M}_S denotes the mapping spaces (in simplicial sets) for the category of EKMM S -modules and S_S^{-1} denotes the canonical cell (-1) -sphere S -module [11, III.2]; ΦX is always a positive Ω -spectrum and when X is a mapping spectrum, ΦX often turns out to be an Ω -spectrum (for example, this happens for $X = \mathcal{F}_{\mathcal{C}^{\text{op}}}(x, y)$ where $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ is the spectral category defined below). The lax monoidal natural transformation is induced by

$$\begin{aligned} \Phi X(m) \wedge \Phi Y(n) &= \mathcal{M}_S((S_S^{-1})^{(m)}, X) \wedge \mathcal{M}_S((S_S^{-1})^{(n)}, Y) \\ &\longrightarrow \mathcal{M}_S((S_S^{-1})^{(m+n)}, X \wedge Y) = \Phi(X \wedge Y)(m+n) \end{aligned}$$

and the map $S \rightarrow \Phi S$ is induced by the map $S^0 \rightarrow \mathcal{M}_S(S, S)$ sending the non-base point to the identity element.

Notation 15.2. For \mathcal{C} a spectral category in EKMM S -modules, write $\Phi\mathcal{C}$ for the associated spectral category in symmetric spectra described above.

Now given \mathcal{C} a spectral category in EKMM S -modules we associate a Waldhausen category to \mathcal{C} as follows. Let $\mathcal{M}_{\mathcal{C}^{\text{op}}}$ denote the category of (right) \mathcal{C} -modules, the category of enriched functors from \mathcal{C}^{op} to the category of EKMM S -modules. We make $\mathcal{M}_{\mathcal{C}^{\text{op}}}$ into a model category with the projective model structure: The weak equivalences and fibrations are the objectwise weak equivalences and fibrations. The cofibrations in this model structure are the retracts of relative cell inclusions, where a cell is of the form

$$\mathcal{C}(-, x) \wedge S_S^q \wedge S_+^{n-1} \longrightarrow \mathcal{C}(-, x) \wedge S_S^q \wedge D_+^n$$

for some object x in \mathcal{C} , $q \in \mathbb{Z}$, $n \geq 0$, where $S^{n-1} \rightarrow D^n$ is the standard n -cell in topology. We then have a subcategory of finite cell \mathcal{C} -modules, having objects the \mathcal{C} -modules built from $*$ by attaching finitely many cells. If we insist on using

canonical pushouts in building these complexes (or restrict to a skeleton), then the resulting subcategory we get is small.

Notation 15.3. For \mathcal{C} a small spectral category in EKMM S -modules, let $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ be the small subcategory of $\mathcal{M}_{\mathcal{C}^{\text{op}}}$ of finite cell \mathcal{C} -modules.

We have a spectrally enriched functor $\mathcal{C} \rightarrow \mathcal{F}_{\mathcal{C}^{\text{op}}}$ sending x to $\mathcal{C}(-, x) \wedge S_S^0$. By the Yoneda lemma

$$\mathcal{F}_{\mathcal{C}^{\text{op}}}(\mathcal{C}(-, x) \wedge S_S^0, \mathcal{C}(-, y) \wedge S_S^0) \cong F_S(S_S^0, \mathcal{C}(x, y) \wedge S_S^0)$$

(where F_S denotes the function S -module) and the map

$$\mathcal{C}(x, y) \longrightarrow F_S(S_S^0, \mathcal{C}(x, y) \wedge S_S^0)$$

is a weak equivalence. The following theorem is now clear from the construction of $\mathcal{F}_{\mathcal{C}^{\text{op}}}$.

Theorem 15.4. *For \mathcal{C} a small spectral category in EKMM S -modules, the spectrally enriched functor $\mathcal{C} \rightarrow \mathcal{F}_{\mathcal{C}^{\text{op}}}$ is a DK-embedding, and $\pi_0 \mathcal{F}_{\mathcal{C}^{\text{op}}}$ is the thick subcategory of $\pi_0 \mathcal{M}_{\mathcal{C}^{\text{op}}}$ generated by the image of \mathcal{C} . In particular, $\mathcal{C} \rightarrow \mathcal{F}_{\mathcal{C}^{\text{op}}}$ is a DK-equivalence if and only if \mathcal{C} is pretriangulated.*

Since $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ is a subcategory of cofibrant objects in a simplicial model category with all objects fibrant, it fits into the context of Example 1.5, and is canonically a simplicially enriched Waldhausen category. In fact, it is easy to see that the tensor in $\mathcal{M}_{\mathcal{C}^{\text{op}}}$ of an object of $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ with a finite simplicial set is isomorphic to an object of $\mathcal{F}_{\mathcal{C}^{\text{op}}}$, so $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ is a simplicially tensored Waldhausen category. The following is the main theorem of this section; combined with the previous theorem, it gives the zigzag of DK-equivalence of spectral categories $\Phi \mathcal{C} \simeq \tilde{\mathcal{F}}_{\mathcal{C}^{\text{op}}}^S$ when \mathcal{C} is pretriangulated.

Theorem 15.5. *For \mathcal{C} a small spectral category in EKMM S -modules, there are zigzags of DK-equivalences of spectral categories (in symmetric spectra in simplicial sets)*

$$\Phi \mathcal{F}_{\mathcal{C}^{\text{op}}} \simeq \mathcal{F}_{\mathcal{C}^{\text{op}}}^S \simeq \tilde{\mathcal{F}}_{\mathcal{C}^{\text{op}}}^S,$$

where $\tilde{\mathcal{F}}_{\mathcal{C}^{\text{op}}}$ denotes the simplicially tensored Waldhausen category constructed from $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ by Definition 14.5.

The zigzag of DK-equivalences $\mathcal{F}_{\mathcal{C}^{\text{op}}}^S \simeq \tilde{\mathcal{F}}_{\mathcal{C}^{\text{op}}}^S$ is the one obtained from applying Theorem 13.9 to the simplicially enriched based weakly exact functor $\tilde{\iota}': \mathcal{F}_{\mathcal{C}^{\text{op}}} \rightarrow \tilde{\mathcal{F}}_{\mathcal{C}^{\text{op}}}$ in part (v) of Theorem 14.1. That leaves us with constructing the zigzag of DK-equivalences $\Phi \mathcal{F}_{\mathcal{C}^{\text{op}}} \simeq \mathcal{F}_{\mathcal{C}^{\text{op}}}^S$, which is just the generalization of Proposition 9.5 to rings with many objects. The proof is essentially identical: Let $\Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}}$ denote the spectral category (in symmetric spectra in simplicial sets) with the same objects as $\mathcal{F}_{\mathcal{C}^{\text{op}}}$, but with mapping spectra $\Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}}(x, y)$ defined by

$$\Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}}(x, y)(n) = \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)),$$

where we have written $F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}$ for the mapping spectrum in $\mathcal{F}_{\mathcal{C}^{\text{op}}}$ to avoid confusion with the mapping space (simplicial set) $\mathcal{F}_{\mathcal{C}^{\text{op}}}(x, y)$. For $y = x$, we have the unit $S \rightarrow \Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}}(x, x)$ induced by the unit for $\mathcal{F}_{\mathcal{C}^{\text{op}}}(x, x)$ and the canonical isomorphism

$$\mathcal{M}_S((S_S^{-1} \wedge S^1)^{(0)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^0 y)) = \mathcal{M}_S(S, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, y)) \cong \mathcal{F}_{\mathcal{C}^{\text{op}}}(x, y).$$

Composition is induced by the smash product map

$$\begin{aligned} \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(m)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(y, \Sigma^m z)) \wedge \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)) \\ \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(m+n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(y, \Sigma^m z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)) \end{aligned}$$

and the composition map

$$\begin{aligned} F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(y, \Sigma^m z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y) &\longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(\Sigma^n y, \Sigma^{m+n} z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y) \\ &\longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^{m+n} z) \end{aligned}$$

analogous to the one in Definition 2.5. We then have spectral functors

$$\Phi \mathcal{F}_{\mathcal{C}^{\text{op}}} \longrightarrow \Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}} \longleftarrow \mathcal{F}_{\mathcal{C}^{\text{op}}}^S$$

defined as follows. The functor $\Phi \mathcal{F}_{\mathcal{C}^{\text{op}}} \rightarrow \Phi' \mathcal{F}_{\mathcal{C}^{\text{op}}}$ is the map

$$\mathcal{M}_S((S_S^{-1})^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, y)) \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y))$$

induced by n -fold suspension

$$F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, y) \longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(\Sigma^n x, \Sigma^n y) \cong \Omega^n F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)$$

and the adjunction

$$\mathcal{M}_S((S_S^{-1})^{(n)}, \Omega^n F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)) \cong \mathcal{M}_S((S_S^{-1})^{(n)} \wedge S^n, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)).$$

The functor $\mathcal{F}_{\mathcal{C}^{\text{op}}}^S \rightarrow \Phi'$ is induced by the map

$$\mathcal{F}_{\mathcal{C}^{\text{op}}}^S(x, \Sigma^n y) = \mathcal{M}_S(S, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y)) \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\text{op}}}}(x, \Sigma^n y))$$

induced by the canonical collapse map $S_S^{-1} \wedge S^1 \rightarrow S$. On mapping spaces, both these functors are weak equivalences (in fact, level equivalences) of symmetric spectra, and so the functors are DK-equivalences. This completes the proof of Theorem 15.5.

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